PROFESSOR: Most likely this will be a two-hour lecture with a short break, so I am teaching until 3 o'clock and we have of the classroom until 3 o'clock. Let's jump right into cold fermions.

Almost everything I say in this talk is some advice in a review paper which Professor [? Swiller ?] and myself wrote a few years ago. The paper will be posted to the group website later this afternoon. OK, when we cool down fermions, Lithium-6 and Boson sodiums we immediately notice a difference. The sodium cloud shrink, shrink, shrinks and forms a small Bose-Einstein condensate, whereas the lithium cloud stops to shrink when we reach the degeneracy temperature.

And, of course, you know it's at low temperature. At high temperature the gases have very similar behavior, but at low temperature they look very, very different. And that can be directly observed. We know that bosons do something very special at low temperature and we talked a lot about it in this course. Namely, they become superfluid and form a Bose-Einstein condensate, whereas, for single-component fermions, there is some interesting physics in form of Fermi sea, but nothing really special. So, in order to do something more interesting with fermions you need two kinds of fermions. And two kinds of fermions can form pairs.

To think, for a moment, two atoms can form a molecule and those Bosonic pairs can condense into Bose-Einstein condensate and becomes superfluid again. If that would be all to Fermionic superfluidity I would be almost done right now and would say, "OK a Bosonic atom is a composite particle made of nuclei and nucleus and electrons. Well, a bosonic molecule is a composite particle, but in the end they do both the same." But the special things about fermions is there is another form of pairing, which is much, much, weaker. And this is the form which leads to

1

superconductivity of electrons. It's Cooper pairing. And this is really much more subtle because the pairs are much larger than the inter-particle separation.

Now, in atomic physics, we have a wonderful tool to study pairing because atoms can form molecules, weakly bound molecules or tightly bound molecules. And we can control that in experiments with the Feshbach resonance. Assume you have two atoms, which collide, and they can form a molecule. And if you apply a magnetic field the two free atoms and the molecule may have a different Zeeman shift, and therefore, there may be a level crossing.

And this is called, as a function of magnetic field, the Feshbach resonance. If the crossing isn't crossing nothing happens because the two configurations would not interact. So, physics happens when there are interactions for some hyperfine coupling, for instance, which takes the crossing into an anti-crossing. And now you see the following physics around this Feshbach resonance, that we have two atoms here. Here, we have a stable molecule.

And there are many, many more molecular states down there, but for this restricted Hilbert space this is the lowest state. It's a stable molecule. The atoms, if you compare the solid line with a dashed line, attract each other because the solid line has lower energy, whereas, on the other side of the Feshbach resonance they repel each other. So therefore, if you look at the scattering lengths of the normalized force between atoms we find this disparity feature around the Feshbach resonance, and atomic physics wouldn't be the same without Feshbach resonances. We can now tune attractive repulsive interactions. In a richer situation there may also be Zero-crossings of the scattering lengths, which allows us to realize non-interacting systems and such. And Feshbach resonances were first observed at MIT some 15 years ago.

So coming back to fermions. This Feshbach resonance gives us something very interesting regarding to pairing. On that side where we have stable molecules, those molecules are bosons and those bosons form a Bose-Einstein condensate of bound pairs. Over here, we have attraction. But the attraction is not strong enough to form

a bound state, at least two atoms in vacuum would not find form a molecular state. The attraction is too weak.

But we know from condensed matter physics textbook if you have many fermions, if you have a Fermi sea, then the Fermi sea plus attraction equals BCS Cooper pairs. So now we have an opportunity, by simply changing the magnetic field, to go from real, strongly-bound pairs to weakly-bound pairs, and eventually to Cooper pairs, which would not form pairs in vacuum. And this is a kind of physics, which was for the first time explored with ultra cold atoms, in the BEC-BCS crossover. Jenny.

AUDIENCE: Just to clarify. So in the repulsive regime, you said the atoms form molecules, but the atoms repel each other. So you're saying atoms that don't form molecules repel each other or [INAUDIBLE]--

PROFESSOR: Yeah. Actually if I go back to that, here they form molecules, here they repel each other. And actually a lot of people think, the repulsive side of the Feshbach resonance means there is repulsion. But look, the atoms attract each other so much that they form a molecule. You can actually see the repulsion of the atoms comes because there is a very attractive molecular state below, and those two states repel each other.

So the repulsion up here is actually caused by the molecular state down there. So in other words, if you go from the attractive to the repulsive side-- many people use it, in terms of pairs-- you go actually from very weak attraction, which is not sufficient to form the bound state, to stronger and stronger attraction, which now forms a stronger, and more strongly bound molecule. Mark.

AUDIENCE: Are there ways you can tell whether the molecules are repulsive, or attractive just by physics, or [INAUDIBLE].

PROFESSOR: You mean what the force is between the molecules now? It turns out that the molecules are repulsive. In some picture here, when the molecule forms, you can see it's a loosely bound molecule. And the repulsion between atoms turns into repulsion of molecules. If you use an [INAUDIBLE] which has approximations, you

would actually find that the scattering lengths for molecules is exactly two times the scattering length for atoms. But if you work it out correctly, the factor is different than 2. Colin.

AUDIENCE: You said that the molecules are boson. I guess they're only bosons if the size of the molecule--

PROFESSOR: I come to that later. We come to that later, yes.

AUDIENCE: My question was, I guess the size of the molecule scales pretty strongly with how far away you from resonance, C6 R6 balance for---

PROFESSOR: In the universal regime, which is open channel dominated-- sorry for talking shop-- in that regime, the size of the molecule is equal to one-half of the scattering lengths. So as the scattering lengths diverges, the molecule is very, very big. And as we move away, the molecule gets smaller and smaller. But once we get out of this quadratic regime, out of the universe of the halo regime, then, of course, the molecular size is completely determined by the close channel. And it is whatever it is in the close channel. And there's no relation to the scattering lengths.

OK. So in other words, it is now possible using Feshbach resonances-- Yes. I'm just missing colors, those dots should be red. Do you have a color problem. The red has turned into black. At least on the monitor. That is much nicer. On the internal monitor it's red, it's just that the computer does not show red. Let me just try that. No, it's not cooperative. Just one more second. I have many more slides with color, so I just want to make sure that you recognize the features. I don't see any easy fix. So maybe we should live with just imagining some of the black is red.

So in other words, by using a magnetic field, we can go from strongly-bound pairs, which form a Bose-Einsten condensate, to weakly-bound pairs, which resemble Cooper pairs of electrons in superconductors. So this is the physics of the BEC-BCS crossover, and in the middle of it we find a new form of superfluidity, which has not really been realized before. Namely, you can see these are molecules which are too big be to be called molecules. Or these Cooper pairs, which are too small to be

4

called Cooper pairs. It's really pairs which sort of elbow each other. They just fill all the space. The pairs are exactly as large as the spacing between atoms.

OK. In class I always try to teach you something conceptional. And the conceptional part I want to address now is, let's assume we are on the right-hand side, on the attractive side of the Feshbach resonance. Two atoms weakly attract each other with their negative scattering lengths, but there is no bound molecular state. So how should you now understand that if you have many fermions, that those fermions form Cooper pairs in pairs?

And the question I would have for you, is-- definitely probably heard about it-- is this paring mechanism, which requires the Fermi sea Is that many-body pairing? That somehow correlation between many particles are necessary, and it's a genuine many body effect? Or can you still understand Cooper pairing as mainly a two-body effect? So in other words, how much many-body is really involved in creating pairs in the presence of a Fermi sea? Does anybody know the answer? What you hear from most people is that it's a real many-body effect. Two particles cannot found form a pair in vacuum. So it really many-body physics, and you say "Wow, I really have to understand complicated physics."

AUDIENCE: [INAUDIBLE]

PROFESSOR: Pardon?

AUDIENCE: [INAUDIBLE]

PROFESSOR: It has something to do with a Fermi sea. What-- [? Timor ?] first.

AUDIENCE: I may know the answer. Maybe you would like to--

PROFESSOR: Take it away.

AUDIENCE: There's always a bound state in two dimensions, I think, between both [INAUDIBLE] the Fermi sea helps you get the two dimensions.

PROFESSOR: That's correct. So what I want to show you is exactly what [? Timor ?] said. You can

really understand Cooper pairs. You can understand the exponential dependence on the scattering lengths by simply looking at one particle, at two particles, with a very weak potential. But you have to look at it in two dimension. And it is eventually the Fermi sea which provides a two-dimensional density of states. So in other words, the binding of Cooper pairs, at least qualitatively, can be understood with a single particle Schrodinger equation with undergraduate physics knowledge. Colin

AUDIENCE: I guess the picture that was sort of when you think of PCS and a real solid, like a three dimensional solid, is that one electron traveling through the lattice repels all the electrons of similar spins [INAUDIBLE] exclusion. And then the electron of another spin moving with opposite momentum, sort of, sees that as a deficit of that spin and sort of attracted to that region. So in that sense isn't, sort of this effective clearing out of other particles to make room for [INAUDIBLE] many bodies associated with Fermi sea [INAUDIBLE]

PROFESSOR: Very good. So we are talking about two different things here. One is we need an attraction. In the atomic system we get an attraction because of van der Waal's molecular forces, that's it. For electrons which have Coulomb charge, where is the attraction? And you wonderfully described that there is attraction between electrons. It's like the couch effect. If you sit on the couch, you indent the couch, and the person next to you will move towards you.

Similarly, when an electron goes through a lattice it leaves a polarization of the lattice behind and the next electron is attracted to this trajectory. So what you are explaining is the phonon-mediated process which leads to attraction. But if I have two fermions which attract each other I don't need this complication.

AUDIENCE: But couldn't you even get it without a lattice [INAUDIBLE] the Fermi gas. If I have one fermion, and I move it, the location that it was has been-- All the other fermions have been pushed away from that so there's a sort of a density depletion there. So couldn't the other spin component see that as effective attraction?

PROFESSOR: Density depletion are density waves, and these are sound waves-- maybe there is a possibility to map it on phonons in the lattice. I haven't heard about it. So I've never

heard about such an attraction which would happen in a Fermi gas, which would give the Coulomb repulsion for some correlation attractive component. What is well-described, of course, is the couch model or the phonon model you mentioned.

But I'm saying, this is just giving you the attraction. But the question is now, and that's the next expression I raised, once we have an attraction between two particles, do we need many-body physics or single-particle physics to have pairing? So let me address that. Now this is out of the review paper, you could be much more about it. But it simply reminds you-- undergraduate problem-- what happens when you have two particles with a very weak attractive potential? And, of course, since you can eliminate the center of mass motion, when I say two particles it's the same as one particle attracted to a central potential.

What you learn, or what could may have already learned in your undergraduate days, is that in 1D, if you have an infinitesimal depths of the square of L, you found a bound state. The bound state will decay very slowly, but it's a bound state. In two-dimension you also find a bound state. In two dimensions logarithm appears. Often when logarithm appears it's sort of the limit. The logarithm is the weakest form of binding.

And if you go one step further, for a very small attractive potential, you do not have a bound state. So in three -dimension you need a minimum. You need a critical energy of the square of L-- critical depths before your bound state-- whereas, infinitesimal attraction is enough to bind particles in one or two-dimension. So therefore, when we talk about fermions with very weak attraction-- in 1D and 2D, no problem. They can form pairs by that, pairs even in vacuum. -- but, in 3D we need something else because an infinitesimal attraction is not enough to form a pair.

Now let me ask you the following question. Let me just cover the slide because it gives the solution-. You all know that in three dimension, when you have radio symmetry, you can transform the three-dimensional Schrodinger equation into an angular equation, but then also into a one-dimensional equation for the radial wave function. And of course, this one-dimensional radial equation has the same potential

as the original three-dimensional equations. So I don't know, I want to hear from one of you. Why does now an infinitesimal square of L potential with attraction give bound states in 1D, but not in 3D? Because in 3D, the radial wave equation uses the same infinitesimal potential as in 1D. Jenny.

AUDIENCE: Because you have to break down an effective potential that takes into account the centrifugal

PROFESSOR: Perfect, but let's use L equals zero. Then the centrifugal correction is zero. Otherwise you are right. It's an effective potential. But let's talk about L equals zero, S states. Then there is no correction due to the centrifugal forces. Colin.

AUDIENCE: Radial derivative is no longer just a simple second derivative [INAUDIBLE].

PROFESSOR: No it is. You have a more complicated radial derivative, but then you go to an equation which just says D2DR squared over function U and it's exactly the one-dimensional Schrodinger equation, which you write down. You can transform the radial equation with a more complicated derivative into something which is exactly a 1D Schrodinger equation. Yes.

AUDIENCE: You can [INAUDIBLE] relationship between [INAUDIBLE]

PROFESSOR: Sure. Uncertainty of relation always can immediately help you if you have an uncertainty relation for for kinetic energy and you realize that you don't have enough potential, but maybe mathematically, what is different between the identically-looking 1D Schrodering equation and the radial Schrodinger equation for the wave function U?

AUDIENCE: I don't know how this is going to lead to it, but I noticed one difference is that in in the three dimensional case you can't go less than zero.

PROFESSOR: Exactly. In the three dimensional case you have a wave function psi, but then you do a substitution that the 1D Schrodinger equation is not for psi, it is for U. And U is the radial part divided by R. So you transform to a different function, which is one more power of R. And the requirement because of this is that your radial wave

function of this one-dimensional radial equation has to have a node in the center.

And now you see what happens. If you start at zero, your wave function shoots up. If you don't have enough attraction it will never curve down and become a normalizable state. Whereas, in 1D you do not have the requirement that the wave function if your 1D Schrodinger equation has a node. So you have the same equation, the same infinitesimal potential, but the radial equation has a different boundary condition. So that's the difference. And that's important. It changes the physics completely.

OK. Now it looks very different and it seems you have to-- Niki.

AUDIENCE: What about 2D?

PROFESSOR: Limiting case, we come to that in a second. In 2D an infinitesimal potential is still enough.

Now I want to show you how you can take care of all dimensions with one equation and not solve three different equations with three different kinetic energy terms. That goes as follows. Recognize that this is just Schrodinger's equation, kinetic energy minus energy equals potential. But now we can Fourier transform it. And the nice thing about Fourier transformation is that the second derivative, spatial derivative, is simply q squared.

So therefore, if you describe now everything in Fourier space, you have q squared plus k squared times psi, but you divide by q squared plus k squared. Here you have a product. The product turns into convolution. Anyway, it may be not the most familiar way you've seen it, but this is just Schrodinger's equation in Fourier components. And now we want to simplify that we have a short-range potential.

So we assume we have a short range potential. If you had a delta function, the momentum components of the potential would be constant for all R. But, we get into something unphysical if you don't put in a cutoff, so we put in a cutoff. But with that cutoff, you can now re-write the equation as follows.

And now we want to do one thing. We want to integrate over q. Then on the right-hand side we also have an integral over q. And then we divide by the common factor. And what you get is this. I know I'm going fast, but this is nothing else than re-writing Schrodinger's equation for you. But now look at it. It has an integral over all energies with the density of states, or of E, of energy. E is k square.

This is the energy of the bound state, and we want the bound state. And on the left hand side we have our infinitesimal potential. And one thing is clear, if you have an infinitesimal potential, we can only get an infinitesimally bound state.

So now we do the following. If the left hand side goes to 0, If you go to an infinitesimal attraction, the left hand side diverges.

And the question is now, if the energy goes to 0, does the right hand side diverge or not? In other words, if this integral, where you said E equals 0, does not diverge, you cannot fulfill this equation with an infinitesimal attraction in the square of l potential. So therefore, the condition for bound state is now that this expression here with E equals 0, diverges. And this combines now all the cases. And you have now an integral equation, and you can now solve it for infinitesimal V0.

You can solve it, what is the energy as a functional of V0? And in two dimensions, when the density of states versus energy is constant, you have the weakest divergence on the right hand side. It's logarithmic divergence. In three dimensions, this doesn't diverge.

In one dimension it diverges big time. Two dimensions is the limit. There's a logarithmic divergence. And if you do the math, which is elementary, but I'm not doing it, you find that the energy of the bound state depends exponentially on minus 1 over the attraction. So for infinitesimal attraction, this turns to E to the minus infinity. It's the weakest bound state you can imagine, and this happens in 2D.

OK, let's go from one particle physics back to fermions. And let me now present to you how Leon Cooper really found the key ingredients to the long standing problem, "How do electrons pair in a superconductor?" And he did the following.

He made an artificial simplification, which you can see is ingenious. He assumed the following. He just wanted to understand, how do two electrons pair? But he simply assumed that those two electrons are on top of a Fermi sea. So he pretty much said, "I want to solve the problem where two electrons just scatter through all available states and then figure out if an infinitesimal attraction between those electrons leads to binding."

Of course, in the real world, the electrons in the Fermi sea are identical to those electrons, to those two special electrons, and they're constantly exchanged. But he simply made this artificial model. And then, of course, we can immediately write down the equivalent equation. It's exactly the same formalism, except for when we integrate over all energies here. We integrate, yes. We integrate over the three-dimensional energy. But now, we integrate only over a small region on top of the Fermi sea.

And everything looks the same. We just have sometimes subtract the Fermi energy. All energies are now measured not relative to 0, but relative to the Fermi energy. If you compare it to the equation I just showed you, forcing the particle bound state, you pretty much see that everything is the same, except for energies are measured relative to the Fermi energy. And secondly, we're not integrating from zero, as before. We start integrating from the Fermi energy.

And that means now, that in this area of integration, the three-dimensional density of states is constant, namely it's the density for where the energy is the Fermi energy and can be pulled out of the integral. And then we have the same logarithmic divergence as we had in two dimensions. So therefore, the problem of two electrons scattering on the surface of the Fermi sea means that we have essentially the same condition for the electrons as if they would live in two dimensions.

Question. If the world were four-dimensional, and we would have the same physics, we have two electrons on the surface of the Fermi sea, would the physics of the two electrons, in that regard, be 4 minus one dimension, three dimension, or would it be

two dimensions?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Well, that's exactly the question. Let's assume you had four spatial dimensions. For this kind of analysis, it's the effective density of states on the Fermi surface. Now, two dimensional? Or is it three-dimensional?

AUDIENCE: It should be three, right? You remove one dimension by restricting yourself the area in the Fermi sea?

PROFESSOR: It is, if you say the particles have momentum on the Fermi surface, if the Fermi surface fixes one momentum, and now they have three different momenta. So in that sense, the motion on the surface in four dimension, is three-dimensional. But, it was sort of a trick question, to some extent, because here we have reformulated everything with energy. And the answer is now, the density of state is constant. It is the four-dimensional density of states, but evaluated at the Fermi energy.

So therefore, no matter in what dimensions you are, because you are restricted to the Fermi sea, you integrate over energy with a constant density of states. And if you pull the constant density state out of the integral you have a logarithmic divergence. And you pretty much get the same answer. So, therefore the physics in four dimension would be the same. It would be the fermions which live on this Fermi surface would just marginally pair.

AUDIENCE: Can you explain the reduction of dimensions? Where does [INAUDIBLE]

PROFESSOR: Well, the motion, you can see on the Fermi surface is n minus one dimension. But the energetic analysis always uses a constant density of state. And the constant density of state is characteristic for two dimensions. So you have both. If you analyze the motion, yes, you have n minus one dimensions. But if you look at the question of bound states, you encounter the same situation as in two dimension, because the effective density of states is constant.

OK so with that, the message I have for you is, Cooper pairing is a single particle

effect. A single particle trapped to an arbitrarily weak potentially in two dimensions. And the only many-body physics which allows those Cooper pairs to form is, that you have a Pauli blocking, that you have Fermi sea, which prevents particles from going into the deep Fermi sea. They stay on the surface, and therefore, they form weak pairs. So the only many body physics, which we need for that kind of pairing is Pauli blocking. Mark.

AUDIENCE: Sorry. Can you explain what is ER in the previous slide?

PROFESSOR: ER was the cutoff. We were cutting off the potential at a short-- we didn't use a delta potential, we used a square of l potential, which extends out to R. And ER is the recoil energy associated with R. Its 1 over R squared, with a few h bars and n to give it the units of energy. Colin.

AUDIENCE: If you go back to that, I can imagine Taylor expanding my density of states in any dimension around Fermi energy. I'll always have my constant term, and the first term is 3D, it's E to 1/2, or some linear term. Doesn't that imply that even if give the system any infinitesimal amount of energy, that integral no longer diverges and--

PROFESSOR: No. The question of divergence is robust. Depends only on the leading term and not on the expansion. If you have a divergence you are not changing it by a small correction term. Because a small correction term is on top of unity. If you start with the density of states at 0, which goes with epsilon to some power, you have a different situation. You catch really the physics by saying you take a constant density of states here.

Anyway, let me just go on and say we should now figure out-- and this is also important when it comes to pairing-- which pairs are the most important ones. And the message I want to give you here, the pairs which are most important one, on the ones which have zero momentum. Because those pairs can scatter to all states on the Fermi surface without violating momentum conservation. However, if you have the total momentum of the two particles which is non-0, they can only scatter on the segment of the sphere. And you got already the message that the higher the density of state, the better it is for the binding. So therefore, the pairs which have

the largest binding energies will be the one with total momentum zero.

OK, now we go from Cooper to BCS. Those three guys figured out how they can now take the physics described by Cooper, and really fold it into a correct many-body formalism. In other words, not to artificially have 2 fermions, which scatter, and there is just the Fermi sea of the other ones. They are now treating the two electrons, which scatter, on equal footing with the whole Fermi sea. So, now we have democracy, all electrons are created equal.

So I want to derive for you the BCS wavefunction in the way which was not the way done by Bardeen, Cooper, Schrieffer, but it makes an immediate analogy with Bose-Einstein condensation. , So what happens is you could now have the idea-- of course I'm very suggestive in leading you to this idea-- two fermions form a pair. And if the pairs both condense, then each pair would be identical.

So maybe we should take our n particles, our n fermions-- n half spin up, n spin down-- each of them should form a pair. So one and two forms a pair, three and four forms a pair, and such. And then we see, our wavefunction of identical fermion pairs slash bosons is just the product of those pairs. But the pairing wavefunction, the molecular wavefunction if you want, so [INAUDIBLE], it's the same for all particles. So this is just like in the Bose-Einstein condensate.

We have a product state of n times the same wavefunction. Here we have a product state of n over two times the same pairing wavefunction. We should anti-symmetrize that's important. Now let us see what we got. Well, we describe everything in second quantization. We use field operators which create particles at a certain position, and then we have the wavefunction of the pairs to make sure they are created in the right position.

We transform everything our operators and do Fourier transform. We go from operators and positions space, to operators in momentum space. But now, it's just formalism. You can read up about it.. But the important thing is now I define a pair creation operator. This pair creation operator creates one of those pairs. And what it does is it creates. We want to focus on the pairs we zero momentum. It forms a

plain wave fermion in the momentum state with minus k, with opposite spin with plus k.

So these are sort of two opposite, plain waves with zero momentum. But now the leading factor is the Fourier transform of this pairing wavefunction. So if you want to assume it's a molecule or wavefunction, Fourier transformers, and this is the leading factor. And with that definition, this is now the creation operator of a molecule consisting of two fermions.

OK. So now, we can say, well maybe a good idea of what describes our system is now, so to speak, a Bose-Einstein condensate, where we create n over two pairs out of the vacuum. Well, first I want to show you that this is actually the BCS wavefunction.

Most people do not to use this form for the BCS wavefunction, but I want to show you mathematically identical the BCS wavefunction. But first I have to clarify is that really a Bose-condensate? If I would have shown you this equation without telling you anything about it, ya, this looks -- n over 2 identical, b, boson-- Bosonic particles. But now we have to be more careful. We have our definition for the pairing operator. And now we can calculate. Are those bosons?

Let's just look at the communicator between b dega, b dega, bb. Yes fine fine. But now, the commutator between b and b dega is not 1. It would be 1 if the probability, that the population of one of those plain waves k, is much, much smaller than 1, then the commutator is full filled.

But the population of momentum states depends on the size of the pair. If the size is huge, you have low lying momentum states, which have a huge population. If the size is small of the pair, you can say, in a tightly-bound system, particles have so much 0 point energy that every given momentum state has pretty much 0 occupation. So in other words, this criterion here, tells us that our pairs are bosons if nk is much smaller than 1.

But if nk is not smaller than 1, then those pairs act out in a Fermionic way. And the

criterion happens exactly when the size of the pairs is comparable to the particle spacing. So in other words, what we have described with this wavefunction is, we have a wavefunction which is a two Bose-Einstein condensate for tightly bound pairs. That's what we expect. It still has this form, but those pairs now feel the Fermi pressure. And therefore, this is automatically taken account in the formalism. Colin

AUDIENCE: [INAUDIBLE]

PROFESSOR: This one.

AUDIENCE: [INAUDIBLE]

PROFESSOR: It's a commutator not the anti-commutator. It's bb dega minus b dega b. I just want to be careful here because, for fermions, you often use the anti-commutator. But I want to check with these bosons, so I use the Bosonic, the commutator, not the anti-commutator.

Let me just show you one thing real quick. Namely, that this wavefunction is identical to the famous BCS wavefunction. Well, almost, not quite. You know that this is a state which has n particles, n over two pairs. Remember when we talked about photons, we had Fock states with n particles, with n photons. But we also had coherent states, where alpha square was n. But this was sort of, the classic coherent state. And this coherent state could be expanded as a sum over Fock states. And one can now say, to some extent, when it comes to certain properties of light, n photons sometimes it's important to distinguish whether it be a Fock state or coherent state. But, for many things, if you have n photons in the same state, you can have a laser, then you have a coherent state, or you have n photons in a cavity. And that may be similar.

So let me show you what happens when I take this Fock state, but construct what is now the coherent state version. In the coherent state version-- this here is a Fock state with j particles-- but now I take a coherent superposition, and I've chosen my factors such that I will get a coherent state. And indeed, I get the coherent state. This is now sort of, what the laser is for Fock state, is this coherent state for a state

with a specific Fermion number.

And this state now, when I just-- you can also say in many-body physics-- we say, I just went from the canonical ensemble, which has a fixed number of atoms to the grand canonical ensemb.e, where the number of atoms fluctuates. But now I can simply use the definitions. If I Taylor expand this component because all higher powers of the Fermionic creation [INAUDIBLE] operators give 0. The Taylor expansion ends after the first term. And with one more step, I have the famous BCS wavefunction.

So therefore the BCS wavefunction, which has been widely used to describe superconductors, is really identical to creating n over 2 pass b, identical pass b, in the vacuum. And I'm just telling you that, because that means now, mathematically, the BCS wavefunction, which many people say describe a very, very different physics, is formally equivalent to the wavefunction for Bose-Einstein condensate.

OK. So what have you done so far? We talked about one particle physics, how very weak attraction leads to bound states. We realized that if fermions live on top of a Fermi sea, it gives at least the binding, the energetic consideration, a two dimension character. And we have two states. And we have bound states for infinitesimal attraction. And now we have used our intuition. If those pairs exists, if they form kind of a Bose-Einstein condensate to construct a trial wavefunction. And now I want to take this trial wavefunction and show you how we can use it to solve the Many-Body Hamiltonian. So far, single particle physics, two particle physics, intuition to get a trial wavefunction, but now we go to many-body problem. Colin.

AUDIENCE: Can you go to the last slide? So the BCS wavefunction has the normal component and the parody component. Where does the normal component come into the Bose--

PROFESSOR: This one? Oh we are talking here about t equals 0. We are talking about a wavefunction. We are not talking about finite temperature where you have distribution over excited states. This is really temperature physics. Was that your question? 100% superfluid fraction. OK, we are back now. [? Timor. ?]

AUDIENCE: Just before you get into this. This maybe a silly question, but why is it that up has to pair with down. It seems ok for up to pair with up, right?

PROFESSOR: No. What happens is, we have assumed here that we have short-range interaction. And up and up can never come close together because of poly exclusion principal.

AUDIENCE: But then in principal, if you have key wave attraction, it could work.

PROFESSOR: You're talking about now, different pairing mechanisms. I'm talking here about the plain vanilla BCS s-wave pairing. And that means spin up and spin down pair. Also, this is where I can work out the analogy with BEC-BCS cross-over and such. So I'm just giving you the simplest example. But yes, in helium-3 at high magnetic field, you have one of the phases where up and up pair for key waves. It's a triplet, superfluid.

OK so we the Many-Body Hamiltonian, and what I want you to sort of, enjoy now is, I took you last class through the bosons. And you should maybe see how the Fermions are different, but we use similar concepts. And so you may ask yourself at the end of the day, "Are the fermions really so different from the bosons when it comes to super fluidity?" But we're getting there.

Let's make the same approximation, we use the delta function here. And we Fourier transform. We go from position space to momentum space. But now comes one important approximation. Fermions are fairly complicated. Look I mean, we have products of four operators and we integrate over three momenta, Initial momenta, and to the initial momentum, the final momentum, and the momentum transfer. So we have an integral over three momenta.

We have to do something, this is too complicated. And the BCS approximation is, the only focus in these sum of pair operators that we scatter, or we have. k prime minus k prime scatters into k minus k. So we pretty much save whenever we have, in the interaction term, two fermions. We are only looking at pairs, where the center of mass momentum is 0. You can say it's an uncontrolled approximation, but it's based on intuition. So the hope is that we still capture the essence of super fluidity by doing this major simplification of the Hamiltonian.

So now I want to show you that this Hamiltonian can be solved in two ways. And this really connects me with things you have learned previously. One is, we can use as a variational ansatz these pair of wavefunctions which I just constructed. That's actually what Bardeen Cooper Schrieffer did, but they used, as a tidal wave function, the BCS wavefunction with the use and [? Weiss, ?] which is identical.

But then I want to show you, partially because I'm feeling nostalgic about the bosons, that you can also use the Bogoliubov transformation. The same method we learned for the bosons. So variation on this just means use this BCS wavefunction, which I just showed you, plug it into the Hamiltonian, and all you do is, you vary the coefficients u and v until you find the minimum of the energy. And this gives you the famous BCS solution. And this is the plain vanilla formalism for superconductors in Fermionic super fluidity. OK I don't have time to go into details, but I think you've got the big picture.

But let me know just emphasize the similarities we had with the bosons. We have a Hamiltonian, even with the BCS approximations that we got writ over 1, sum over momenta by focusing on the pairs with c [INAUDIBLE] mass momentum. You still have products of four operators. We want to get rid of it. But now we use a mean field idea. We always get rid of products of four operators by grabbing one or two of them, and saying, "we don't look at the operator, we look at the expectation value, which is the mean field." So we want to make a mean field approximation.

But when you go for bosons more simply, you just took the operators, c zero, zero momentum, and replacing with the square root of n 0, the number of atoms in the condensate. But now we have to be a little bit more subtle. So what we should now define, is a pairing mean field. Not an expectation value of c 0, but an expectation value of this creational annihilation operators four pairs. A little bit more abstract, but it follows the same logic.

And then we do the decoupling approximation, which I lectured about to you with a [? mod ?] insulator, that whenever we have products of operators, of products of pair operators now, we take the mean value, but we neglect the product of the

fluctuations. Exactly what I've shown you before when we described the [? mod ?] insulator to superfluid transition.

OK. This is the pairing field with an index k and eventually it's important to define a delta. The famous gap in the BCS, with the BCS COE, which is a sum over all pairing fields with k. And it is this delta which plays the role of the condensate wavefunction exactly the same way as psi, the macroscopic wavefunction in the [? Kospiesky ?] equation [INAUDIBLE]. So you do the same thing, we just write it down, we drop quadratic terms, which are fluctuation terms.

And the moment we have now, simply bi-linear product of operators. We do exactly the same as we did with bosons, we transform to a new set of operators, which is the Bogoliubov Transformation. Of course, this time, we want to make sure that those transforming Fermionic operators, we want to make sure that after transformation we fulfill Fermionic commutator. And then we determine coefficients in such a way that cross-terms cancel out, and what we obtain now is a diagonalized Hamiltonian.

It's diagonalized, it has the ground state energy. And this, you now realize, this is nothing like an harmonic oscillator. It's a quasi particle gas of independent excitations. Gamma k is an excitation with energy E K, and this means we've diagonalized the problem.

I can't do justice to the formalism. I think the best I can do here for you is to show how the same ideas come back and solve the same problem. Also everything is now for fermions. Mark.

AUDIENCE: [INAUDIBLE] Separate to [INAUDIBLE] At the very top of the line, you have [INAUDIBLE] then the next slide, you have the delta, which is the sum over k expectation [INAUDIBLE] sum.

PROFESSOR: I derived it myself because it confused me. But I can just reassure you. If you take the original Hamiltonian, pluck this into here, and neglect the product of the smaller fluctuations, you get exactly this equation. It's really just accounting and regrouping

terms. There is no assumption, no concept involved, it's really just plugging it in. The physics is that we neglect the correlation of the fluctuations.

OK. Yeah, I think I want to wrap up the fermions. We still want to talk a little bit about ions, but we have now solved it. Either with the a variational [? onsets ?] or with the Bogoliubov Transformation. Two very different ways lead to the same solution. We have quasi particles. Here, we determined what the energies are. They are a little bit complicated equations to find what the quasi-particle energies are. But everything is known, we can now throw in temperature. Colin wanted final temperature. Well what happens with this Hamiltonian final temperature? Well, we just have occupation number for those quasi-particles, the thermal occupation, and we have to self-consistency.

Now look what happens in the presence of quasi particles. This eventually leads to an equation for the critical temperature. So what we see here is a function of the inverse scattering lengths. Here we have a BEC temperature which is constant. And when we go on the BCS side, where we no longer form bound molecules, we have is exponentially decaying temperature for very small attraction. The transition temperature to the Superfund superfluid state is exponentially small.

All this is theory, but it has been experimentally observed. If you want to pair fermions, we just need two states. At MIT we use mainly Lithium-6. Lithium-6 has many hyper find states, but we just use the two lowest one. In a high magnetic field, where we have a Feshbach resonance, we pair the particles-- well we needed the lasers and equipment-- and then we observe pair condensates. They look exactly like atomic condensates, but their pairs inside. That they're really pairs inside we found out with a f spectroscopy. And eventually when you locate the gases we find vortices. And you know already, vortices are the smoking gun of super fluidity.

So with that I've rushed you through an important and very elegant chapter of physics, the BEC BCS cross-over. Let me just sort of end with an outlook that I've talked about basic phenomenon in bosons and fermions, both centered about super fluidity, because I think it was pedagogical to show you the differences and the

similarities. But we can now regard those ultra-cold bosons and fermions as building blocks of quantum simulators, building blocks to understand, in its simplest manifestation, interesting physics. So you can see, the Bosic gases are quantum simulator for superfluid helium. The Fermi gases are a quantum simulator for superconductors. We briefly talked about optical lattices, which can be regarded as a [? quantum ?] simulator for crystalline materials. But there is much more you can do with those building blocks.