# Assignment #3

Due: Monday, March 11, 2011

1. **Lossy interferometer**. A real interferometer usually suffers from loss of photons in its arms, or due to imperfect beamsplitters. The effect of such loss can be modeled by a beamsplitter, as we explore in this problem.

   a) Our starting point is the standard Mach-Zehnder interferometer, which uses two 50/50 beamsplitters, and has a phase shifter of variable delay ϕ:
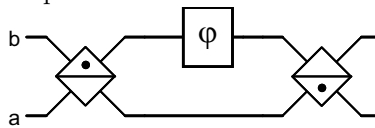

   In the absence of loss, and given an input $|10\rangle$ (notation $|ba\rangle$), a photon is measured to be output in mode b with probability $P_b = \cos^2(\phi/2)$. The *fringe visibility* of this output signal is defined as

 $$V = \frac{\max(P_b) - \min(P_b)}{\max(P_b) + \min(P_b)}, \tag{1}$$

   where the min and max are taken over ϕ. Show that $V = 1$ for this ideal situation.

   b) Suppose now that one arm of the interferometer is lossy, and we model this loss with a beamsplitter, in this way:
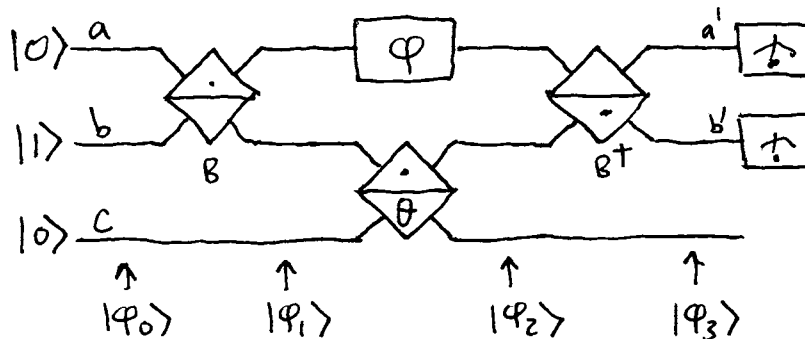
   Let the input state be $|\phi_0\rangle = |010\rangle$ (using notation $|cba\rangle$ for the three modes). The two outer beamsplitters are the usual 50/50 beamsplitters of the Mach-Zehnder interferometer, while the new middle beamsplitter has angle θ. When $\theta = 0$, no photons are lost from b into c, and when $\theta = \pi/2$, any photon in b is moved into mode c, modeling complete loss. Give the intermediate states $|\phi_1\rangle$ through $|\phi_3\rangle$, and compute the probability that a photon is found at the output in mode b, as a function of ϕ and θ. What is the fringe visibility as a function of θ?

c) Now let both arms of the interferometer be lossy, modeled with two beamsplitters, in this way:
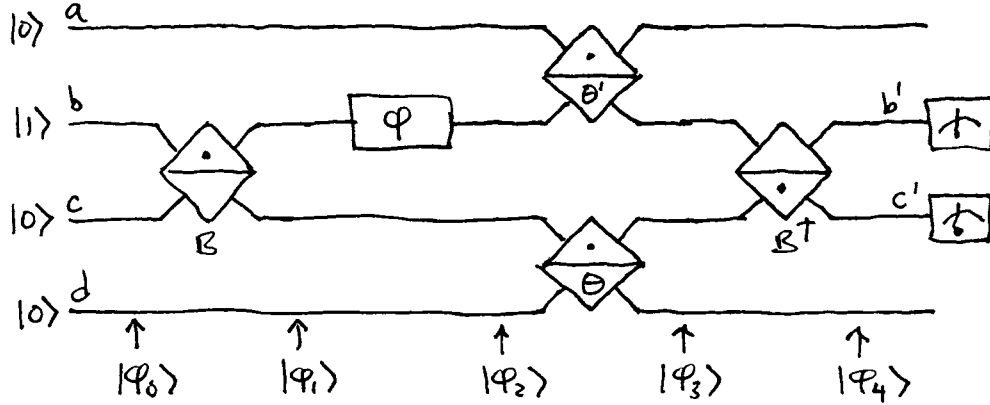
The top beamsplitter has angle θ', and the bottom one θ. Give the intermediate states $|\phi_1\rangle$ through $|\phi_4\rangle$, and compute the probability that a photon is found at the output in mode b, as a function of ϕ and $\Delta\theta = \theta - \theta'$. What is the fringe visibility as a function of $\Delta\theta$? Suppose you include only cases when at least one photon is measured to be output in b or c (in other words, throw away the case when no photons are measured at the output). How does the visibility change, when using this conditional probability?

The case when $\Delta\theta = 0$ is known as the *balanced loss* Mach-Zehnder interferometer, and your results should show why balancing the loss in the two arms is helpful. Restricting cases to when at least one photon is detected implements a simple kind of quantum error detection code!

d) In the scenarios above, a single photon input was used. Do any of the visibilities change if a coherent state were input instead? If not, then describe a situation in which the effects of loss would distinguish interferometers used with single photons versus coherent state inputs.

2. **Generation of Squeezed States by Two-Photon Interactions**

Consider a mode $(\vec{k}, \vec{\varepsilon})$ with wavevector $\vec{k}$ and polarization vector $\vec{\varepsilon}$ of the electromagnetic field with frequency ω whose Hamiltonian H is given by

$$H = \hbar\omega a^\dagger a + i\hbar\Lambda \left((a^\dagger)^2 e^{-2i\omega t} - a^2 e^{2i\omega t}\right) \tag{2}$$

where $a^\dagger$ and a are the creation and annihilation operators of the mode.

The first term of Eq. 2 is the energy of the mode for the free field. The second term describes a two-photon interaction process such as parametric amplification (a classical wave of frequency 2ω generating two photons with frequency ω). Λ is a real quantity characterizing the strength of the interaction.

(a) Write the equation of motion for $a(t)$ Using the Heisenberg picture. Take

$$a(t) = b(t)e^{-i\omega t}. \tag{3}$$

What are the equations of motion for $b(t)$ and $b^\dagger(t)$?

(b) Using the Heisenberg picture, the contribution of the mode $(\vec{k}, \vec{\varepsilon})$ to the electric field is written

$$\vec{E}(\vec{r},t) = i\mathcal{E}_\omega \vec{\varepsilon} \left(a(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger(t)e^{-i\vec{k}\cdot\vec{r}} \right) \tag{4}$$

where $a(t)$ is the solution of Eq. 3. Show that

$$b_P(t) = \frac{b(t) + b^\dagger(t)}{2} \qquad \text{and} \qquad b_Q(t) = \frac{b(t) - b^\dagger(t)}{2i} \tag{5}$$

(where b(t) is defined Eq. 3) represent physically two quadrature components of the field. Find the equations of motion of $b_P(t)$ and $b_Q(t)$ and give their solutions, assuming that $b_P(0)$ and $b_Q(0)$ are known. Give the corresponding solutions for $b(t)$ and $b^\dagger(t)$.

(c) Assume that at $t = 0$, the electromagnetic field is in the vacuum state. Calculate at time t the mean number of photons $\langle N \rangle$ in the mode $(\vec{k}, \vec{\varepsilon})$ as well as the dispersion $\Delta b_P(t)$ and $\Delta b_Q(t)$ on the two quadrature components of the field. Explain the results.

3. **Noise properties of squeezed states.**

We now look at noise properties of squeezed states generated in question 2. The results obtained will be useful here, namely, the transformations under squeezing of b and $b^\dagger$, of the quadrature components b_P and b_Q and of their dispersions Δb_P and Δb_Q.

We again start with a vacuum state at $t = 0$ and, for simplicity, fix the interaction time to $t = t_0$. Let us define the squeezing parameter of the resulting squeezed vacuum state $r = 2\Lambda r$.

(a) Calculate the photon number variance $< \Delta n^2 >$ of this squeezed vacuum state. How is it related to the variance of the two quadrature components of the field $< \Delta b_P^2 >$ and $< \Delta b_Q^2 >$? Take the limit of large squeezing $r >> 1$. How does the photon number variance scale with the mean photon number of the squeezed vacuum state? Compare this scaling to a strong coherent state (classical laser beam).

(b) Consider now that our squeezed vacuum state is attenuated, and as in problem 1, we model the loss with a unitary beam splitter with vacuum state input at the second port. Let $T = t^2$ be the transmission coefficient of the beam splitter (where t is the amplitude transmission coefficient) and $R = 1 - T$ is the reflection coefficient. Let us quantify the quality of the squeezed state by the ratio c^2 of variances of the two quadratures (larger variance over smaller variance). At what attenuation (value of T) is the quality of squeezing c^2 reduced by a factor of 2?

(c) Now let us see what happens to the squeezed vacuum state when we replace the vacuum state input at the second port of the beam splitter with a coherent state $|\alpha >$ (local oscillator (LO)). Remember that mixing with a coherent state on a beam splitter approximates a displacement operation. Assume that the coherent state displaces the squeezed vacuum along the lower-variance quadrature (this corresponds to a choice of phase of α). Sketch the phase space of the output state.

    i. What is the mean photon number at the output? Express the result in terms of the mean photon numbers of the input states.

ii. What is the photon number variance at the output? (Hint: remember from class notes, that the squeezed vacuum state is composed of even-photon-number eigenstates only.) Identify the different contributions. Take the limit of large squeezing and small attenuation (such that it does not degrade the quality of squeezing - refer to part (b)).

iii. Can sub-Poissonian photon number fluctuations be achieved at the output? If so, what are the constraints on the strength of the LO $|\alpha|^2$? Can you give intuitive reasoning for why mixing with the LO changes the nature of photon number fluctuations of the squeezed state? This goes to show that squeezing can be measured on a single photo-detector, without the need to measure field quadratures.

4. **Teleportation**

The two-mode state $|\Psi_{EPR}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ has many interesting properties, one of which is its usefulness in "teleporting" unknown quantum states, as explored in this problem.

a) Let the two modes of $|\Psi_{EPR}\rangle$ be separated by a considerable spatial distance, such that Alice has the first photon, and Bob has the second. Alice is also given an unknown quantum state $|\psi\rangle = a|0\rangle + b|1\rangle$, so we may write the complete state of all three modes as

$$|\psi_0\rangle = |\psi\rangle|\Psi_{EPR}\rangle \tag{6}$$

$$= \left[[a|0\rangle + b|1\rangle] \right] \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{7}$$

$$= \frac{a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle}{\sqrt{2}} , \tag{8}$$

where the first two modes are Alice's, and the third is Bob's. Suppose that Alice can measure her two modes in the basis defined by the four orthogonal states (the "Bell state basis")

$$|\phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{9}$$

$$|\phi_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{10}$$

$$|\phi_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \tag{11}$$

$$|\phi_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} . \tag{12}$$

What single-mode state does Bob end up with, for each of Alice's four possible measurement results? Show that if Alice sends Bob two bits of information, he can use that to perform a corresponding unitary operation on his state to recover $|\psi\rangle$.

The rest is for your interest only. The Bell basis measurement used in the scheme above is difficult to accomplish with optics, and generally requires some kind of nonlinearity that is challenging to realize with single photons. However, an alternate teleportation scheme can be employed, which uses squeezed states instead of photon number states, allowing

a beamsplitter and homodyne measurement to accomplish the Bell basis measurement. This can work as follows. Let

$$|\phi_P\rangle = \frac{1}{\mathcal{N}(P)} \int_{-\infty}^{\infty} d\alpha_x \; e^{i\alpha_x P}|\alpha_x + iP\rangle, \tag{13}$$

where $|\alpha_x+iP\rangle$ denotes a coherent state, and $\mathcal{N}(p)$ is some normalization factor which we may neglect. This is an infinitely squeezed state with well defined momentum quadrature value P (convince yourself of this by sketching its $Q(\alpha)$ plot). Similarly,

$$|\phi_X\rangle = \frac{1}{\mathcal{N}(X)} \int_{-\infty}^{\infty} d\alpha_p \; e^{-iX\alpha_p}|X + i\alpha_p\rangle \tag{14}$$

is an infinitely squeezed state with well defined position quadrature value X.

Recall that a 50/50 beamsplitter U transforms the two-coherent state input $|\alpha\rangle_1|\beta\rangle_2$ as

$$U|\alpha\rangle_1|\beta\rangle_2 = \left|\frac{\alpha + \beta}{\sqrt{2}}\right\rangle_1 \left|\frac{\beta - \alpha}{\sqrt{2}}\right\rangle_2. \tag{15}$$

Suppose that the two output modes of the beamsplitter are measured in the $x \sim a + a^\dagger$ and $p \sim a - a^\dagger$ quadratures (modes 1 and 2, respectively), giving measurement results X and P. By calculating $U^\dagger$ acting on $|\phi_X\rangle_1|\phi_P\rangle_2$, it can be shown that the quadrature measurement projects an arbitrary input state into the basis defined by this (unnormalized) state:

$$|\phi_{XP}\rangle = \int_{-\infty}^{\infty} d\alpha_x \int_{-\infty}^{\infty} d\alpha_p \; e^{i(\alpha_x P - X\alpha_p)} \left|\frac{-\alpha_x + i\alpha_p + X - iP}{\sqrt{2}}\right\rangle_1 \left|\frac{\alpha_x + i\alpha_p + iP + X}{\sqrt{2}}\right\rangle_2. \tag{16}$$

These basis states define a continuous variable Bell basis, where X and P are analogues of the two indices describing the four discrete Bell states of Eqs.(9-12).

We have shown that a beamsplitter and quadrature measurements can be used to perform a kind of Bell basis measurement. To implement teleportation, we also need an analogue to the entangled state $|\Psi_{EPR}\rangle$. Specifically, we would like one of the basis states $|\phi_{XP}\rangle$; for example, it would be satisfactory to have

$$|\phi_{XP=00}\rangle = \int_{-\infty}^{\infty} d\alpha_x \int_{-\infty}^{\infty} d\alpha_p \left|\frac{\alpha_x + i\alpha_p}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha_x - i\alpha_p}{\sqrt{2}}\right\rangle_3, \tag{17}$$

where we have labeled the modes by "2" and "3" for later convenience. This state is a two-mode squeezed state. Specifically, consider the two-mode squeezed state

$$|\Psi_{EPR}^s\rangle_{23} \propto \sum_n s^n|n\rangle_2|n\rangle_3. \tag{18}$$

Such a state can be generated in the laboratory by a non-degenerate optical parametric oscillator. It may be rewritten in the coherent-state representation as

$$|\Psi_{EPR}^g\rangle_{23} \propto \int_C d^2\alpha \; e^{-|\alpha|^2/g^2}|\alpha\rangle_2|\alpha^*\rangle_3 \tag{19}$$

where the integral is taken over the complex plane. It is easily found that $g = \sqrt{s/(1-s)}$.

For more info on the teleportation protocol, refer to the AMO Wiki: https://cua-admin.mit.edu/apwiki/wiki/Non-classical_states_of_light
and to the experiment carried out in Jeff Kimble's group at Caltech (Furusawa et al, Science vol. 282, p. 706, 1998).

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