PROFESSOR: Let's get started. We want to wrap up today our discussion of atoms without external field, and then discuss what happens when we put atoms into external magnetic fields. Last class, we talked about isotope shifts, and there was a question, how big are those isotopes for some of your favorite atoms?

I just looked up some information on lithium, which is a light atom, and this paper here shows calculations compared with experiments. The isotope shift between lithium six and lithium seven due to the mass, lithium six to lithium seven, is about 10 gigahertz. The volume effect is only two megahertz, 1,000 times smaller. However, the precision of experiments is such that if you find an isotope shift, the mass effect can be exactly calculated from the atomic masses, you can still get information about the size of the atomic nucleus out of it. So this is the example of a light atom, 10 gigahertz mass effect, two megahertz volume effect.

And here is a rubidium atom. The isotope shift of the D1 line between 85 and 87 is 77 megahertz only. The mass shift is 56 megahertz, and the remainder is mainly the volume effect. There's a specific mass effect due to electronic correlations if you calculate electrons, which I think is small here but I don't want to discuss it.

We can now compare the mass effect in rubidium versus lithium. First of all, the shift compared to infinite mass. the reduced mass effect, is much bigger in lithium. It's 20 times bigger in lithium than in rubidium. However, when you go from lithium six to lithium seven, the mass changes by 15%, so the delta m over m is also much larger for lithium than rubidium. And if you add those two factors, you find that the mass effect is 200 times larger in lithium, 10 gigahertz versus 50 megahertz. The nuclear volume effect is two megahertz for lithium, 20 megahertz for rubidium, so that sets the scale. Any questions about that?

Let me come back to one other question which we discussed last class, and this was the question about if you have a deformation of a nucleus, or if you have any kind of isotropic shape of an object, what is the minimum angular momentum in order to observe it? I know a lot of you got confused about it, so I want to discuss the same thing again, but now focusing on two different frames, the lab frame and the body-fixed frame. I hope you find this discussion insightful.

So let's assume we have an object, it can be a molecule. Actually, you have a homework assignment whether you can observe a permanent dipole moment of a molecule, and this will lead you into a discussion of lab frame versus body-fixed frame. But let's assume we have an object, assume it's a nucleus, which has a really odd shape. However, if it has angular momentum of zero, all you have is one level. If you have an angular momentum of 1/2, you have two levels, and you can now define in the laboratory that the energy difference is due to the magnetic dipole moment, for instance, or the electric dipole moment if you put in electric field.

If you have I equals 1, you can have three levels, E1, E0, E minus 1. Let's say you have put the atoms into an electric field gradient. You can then ask if E0 is in the middle between up and down, plus 1 or minus 1, or whether it's displaced or not. And depending whether this is larger or smaller than zero, you would say there is a quadrupole moment which is larger or smaller than zero.

In other words, what I'm telling you is if you have only one level, I equals 0, you can't say anything about the shape of the object. If you have two levels, you can determine a dipole moment. If you have three levels, by the deviation from the equidistance, you can find a higher order moment.

But now comes the point. You can now take two positions. You can say that for low I, the deformation of a magnetic moment for I equals 0 cannot be measured. Therefore, it is zero. Or you can say deformation exists, but only not in the lab frame but in the body-fixed frame.

So in other words, you would say that the deformation exists always. It's just a

measurement problem. At low angular momentum, I cannot measure. Or the other statement would be, well, if you can't measure it, [INAUDIBLE], it means it doesn't exist. So which statement or which conclusion is correct? This is the body-fixed frame argument, and this is more based on the lab frame.

Well, I would say the lab frame argument is always correct because you can't measure it, you can't determine it in the lab, but let me go from there. Let's assume we have an object, and we think it has a deformation but it has a low angular momentum phase. Can we still say that in its body-fixed frame, we have an object which has a deformation or not?

Now, my personal opinion is the following. It really depends. If you have a system where you can have angular momentum without changing the internal structure. For instance, I have this stick, and I can just spin it up, and if I can spin it up, I can get an angular momentum wave packet, I can orient it and measure its deformation. Then I think I would say, even if this state has zero angular momentum, it has a deformation, and the way I know it because if I add angular momentum, I can measure the deformation. I cannot measure it at low angular momentum. So then you would say in the body-fixed system, there is a deformation, but it only manifests itself in the lab frame if I add angular momentum.

However, you may have an object, let's say a molecule, which is so weakly bound that one quantum of angular momentum due to centrifugal forces rips it apart. That exists. Extremely bound state which cannot provide enough binding force to withstand even one unit of angular momentum. So that's an object which you cannot rotate, you cannot transfer any angular momentum without destroying it, without ripping it apart. And now to say that this molecule has a dipole moment or has some anisotropy, you're making a statement which cannot be tested at all. So at that point, you should rather say, what matters is what I can measure in the lab, and I will never be able to measure any deformation in the lab.

So the first example with the stick will apply to a very stable molecule which may have a dipole moment, and you assume it has the same dipole moment whether it's

rotating or not. And then you would say at zero angular momentum, I cannot measure the dipole moment but I know it exists in the body-fixed frame.

If you have a nucleus, and you have sort of a wave function of the protons and neutrons, and it's I equals 0, you will not find any moment. At least for the ground state of nuclei, when you add angular momentum, you really change the internal structure. You have to promote nucleons to higher orbits, so you cannot add angular momentum and still have the same object. So therefore, you have to say the ground state with I equals 0 has no deformation, because there is no way to ever find [INAUDIBLE] deformation, so it doesn't exist.

But there are excited states of nuclei which have a deformation, and these deformed nuclei can be put into a multiplex of angular momentum states. So then you would see, I have the same kind of nucleus, but at different angular momentum states. At higher angular momentum states, you can define [INAUDIBLE] determine quadrupole moments and things like this. And then you may say the same internal state has now a non-rotating state, and you would still be tempted, and you are correct with that, to associate a deformation even in the non-rotating state.

I hope those remarks help you reconcile the two aspects whether you have an object which is stable enough to be spun up, and then I think you can always talk about the body-fixed frame. But if you have an object where you change the internal structure when you add angular momentum, I think, for fundamental reasons, you cannot associate any deformation with it. Any questions about that?

Let me just write down the summary. The definition of a deformation in a body-fixed frame makes sense only if you can add or change angular momentum without significantly changing the internal structure. Questions? I think the computer is set up. I cannot go backward in Presentation mode.

What I want to discuss next is give you a little bit of an historic summary how spectroscopy of hydrogen is developed, in particular also focusing on one important discovery, the discovery of QED through the Lamb shift. Of course, you all love hydrogen because it's the simplest atom but it has so much interesting physics in it.

I've summarized for you here some papers on hydrogen, and I used them to illustrate several points.

I will show you that actually, the discovery of the Lamb shift had precursors. 10 years before the Lamb shift was discovered, people had even some idea that something may be wrong with the understanding of the structure of hydrogen. So you can say they came so close, people 10 years before, in realizing the Lamb shift, there were people who maybe missed the Nobel prize by just a tiny little bit. They had all the insight that there may be QED correction in hydrogen. They just didn't have the technology to measure it accurately enough.

The second example I want to show is that we always talk about fundamental limitations, but fundamental limitations can disappear in time because they may not be as fundamental as they appear. So for instance, there were limitations with the limitations of the Lamb shift because you had a short lifetime of p states, but with the event of two photon transition, you can go from s to s and s to d states, and therefore map out Lamb shifts with much, much higher precision not limited by the finite lifetime of p states.

Finally, you would say Lamb shifts are small splittings, and for many, many years, Lamb shifts were measured by making radio frequency transitions between two s and two p states. Well, today, the most accurate measurement of the Lamb shift is with an optical transition where you need a much, much higher relative precision to see the tiny Lamb shift. But optical metrology with direct frequency measurements and frequency combs has so much improved in precision that now, an optical measurement, even if it comes to a tiny difference, is more accurate than a direct [INAUDIBLE].

So I think the history of hydrogen shows you that technology can completely change the paradigm how measurements are made. Fundamental limits disappear because new tools or new insight is available. And also, I find it interesting that discoveries often have precursors, and people have a hunch, know about it, and then finally, it is discovered. Let me just take you through some historic papers.

This paper is 1933, 15 years before the Lamb shift, and it says that one possible explanation for some discrepancy of the structure of the Balmer lines is that the effect of the interaction between the radiation field and the atom has been neglected. That's QED. You cannot just calculate the structure of the hydrogen atom from the Coulomb field. You have to allow the radiation field, all the modes of the vacuum to be included. So this insight is not due to Lamb It was there already, 1933.

Same year, look at the title, "On the Breakdown of the Coulomb Law for the Hydrogen Atom." People speculated or discussed that the Coulomb law will not be valid at very small distances. This is ultimately what QED, raided correction, the Lamb shift, vacuum polarization all is about.

Finally, people had an understanding of the hydrogen atom, and they measure-- I want you to keep that in mind-- they did optical. They measured the Balmer lines of hydrogen and deuterium, and they couldn't fully resolve it because of the finite lifetime of the peak state, but there was some hunch when you try to get the envelope from the underlying structure that there was a discrepancy. It was just not significant enough to say for sure, there is an additional line shift which is not accounted for by theory.

There was a discussion that there is a deviation of the Coulomb law, but here is the insight. As was indicated by previous authors, the interaction required to change the Coulomb law at small distances is much too large to be accounted for by the assumption of a finite size of electron and proton. So the Coulomb field has to be modified at short distances in a much stronger way than just the finite size of the proton. We'll talk about the finite size of the proton in a minute.

And then, of course, 1947, UIF oscillators have been developed in the pursuit of radar, experimental tools are there now-- high power IF sources, cumulative sources, and such, and then Lamb and Retherford in his landmark paper look at the fine structure of the hydrogen atom, and this is the famous result. They measured transitions as a function of magnetic field, and you see the solid line, which I think

was the theory without the Lamb shift. The dashed line is the hyper fine structure of hydrogen, and the lines converge, and the difference is 1,000 megacycles, the first determination of the Lamb shift.

So this was 1947. It's interesting that it's just one or two weeks later, there is a theoretical paper by Hans Bethe providing an explanation for the Lamb shift, so already coming up with the first model how to account for QED. I didn't look it up in detail, but I thought the spirit was similar to what I presented you in class, that the electromagnetic zero point energy is shaping the electron and leading to corrections. But it's amazing that within weeks, theorists figured nearly out, yes, this is the explanation. This is how we have to explain the theory.

For a number of years, people pursued measurements of the Lamb shifts with higher and higher accuracy. I just like the last sentence here. This is now the next paper by Willis Lamb and Retherford, and they are sort of saying that they wanted to measure the Lamb shift with higher precision, but then they said, "the program was large and encountered unexpected difficulties which required much more time to surmount. As a result, the paper promised two years ago was delayed." I think this applies to many, many papers to be written, but here, the authors even say that upfront, it took us two years longer to do the research than we initially anticipated.

You see now the growing accuracy. We have the Lamb shift, which is on the order of 1,000 megacycles, one gigahertz, and the precision is now in the 100 kilohertz range. This was the technology of the original discovery. Then there was a next generation of experiments on the Lamb shift using separated [? oscilloatomic ?] fields, [INAUDIBLE] techniques. We'll talk about that later in the course. And with these techniques, the accuracy of the Lamb shift is now one digit further in the 10 kilohertz region.

There is a nice feature that we'll also discuss later, that it was possible to obtain line weights, which is [INAUDIBLE]. We'll talk about it later, but it's possible to do spectroscopy on unstable states, which provides line widths narrower than the actual line widths. If you want to get one sentence as an appetizer, you just look at

the atoms which have not decayed for a long time, and if you play some tricks, you can then get line widths which covers points to several lifetimes, and not just to the one [INAUDIBLE] lifetime. Some conditions have to be met, and those authors used it here to advantage to narrow the line for the measurement of the Lamb shift. Yes?

AUDIENCE: The abstract says that the result is not in good agreement with theory. What is the theoretical-- how far had it gone?

PROFESSOR: I don't know it at this point. I will later give you comparison with theory which is highly accurate. I assume at this point-- I'm not sure if it was instrumental difficulty. I didn't [INAUDIBLE]. So these are the same authors just a few years later. The agreement between theory and experiment is two standard deviations, so I think this problem disappeared. I don't know if it was the fault of theory or experiment.

But now we go a step forward to optical spectroscopy. You remember originally, and this is why the Lamb shift was not jumping into people's eye when they did spectroscopy of the Balmer spectrum of hydrogen, you cannot resolve the structure. The Lamb shift is there, but it's hidden in the envelope of the unresolved lines.

And now the advent of lasers, and people immediately developed saturation spectroscopy. That's how most of our laboratories stabilize lasers using Doppler-free saturation spectroscopy. And when saturation spectroscopy was invented by Hansch and Schawlow for the first time, you can break through the Doppler [INAUDIBLE], and now you see the lines resolved. And here, I think for the first time, you see two peaks separated and the splitting is the Lamb shift, which until then was only accessible through [INAUDIBLE] frequency methods.

Of course, these were the first lasers, just pulsed lasers, and we couldn't even think about precision. But then, of course, using metrology, using frequency chains, people could do precision measurements in the optical domain. These are now papers in the '90s, "Optical Measurement on the Lamb Shift in the Ground State or in the Excited State."

Talking about the comparison with experiment and theory, experimental is pushed

to higher and higher precision, and suddenly, there was a discrepancy, and it was a discrepancy in the 1s Lamb shift. In the 1s state, the Lamb shift is much bigger than in the 2s state because the electron interacts much more intimately with the [INAUDIBLE] than the Coulomb potential is on this. People found that the experiment did no longer agree with theory.

But then the theorists had to check all their assumptions, and it was found that there were two new binding corrections which were surprisingly large. Often, you make an estimate that those terms are small. You say it's higher order, but you may not know the pre-factor. And here, something was surprisingly large, and by now you're proving the theory. There was again agreement between experiment and theory. It's getting now down to the kilohertz level.

And at least as a few years ago, this was state of the art. Remember the Lamb shift is about 1,000 cycles one gigahertz, and now the precision is in the single kilohertz. As I pointed out, precision was reached by directly measuring the frequency of laser, frequency metrology.

This was actually, for historic interest, frequency metrology, they used peak nodes between the laser used to measure the hydrogen line and some other lasers. This is just a few years before comb generators. Frequency combs completely changed things again. But they had already the precision of a direct frequency measurement. You can read about it when I paused it. It shows you what the [INAUDIBLE] is in those installations.

This is a slide I borrowed from Ted Hansch, "Optical Spectroscopy of Hydrogen." It just shows the advances in frequency metrology. It shows how caesium clocks and optical spectroscopy, how they have changed in precision, and eventually, we are now past the time where optical spectroscopy is more accurate than microwave and radio frequency spectroscopy. Optical clocks are more accurate than [INAUDIBLE] frequency standards. I'm sure in your lifetime, you will experience the redefinition of this again because the caesium clock is no longer accurate enough compared with the most precise optical measurements.

And I think there was a peak gap, a change in slope here. In the old days, you measured the wavelengths of light by making a measurement of the wavelengths using maybe a grading or interferometry, but when you now started to measure frequencies directly, then there was a change in slope and major improvements in precision. And today, of course, the most precise measurement of laser frequencies is not through the wavelengths, that they can count the number of cycles in a peak node with an optical comb generator.

So what can you do with ever increasing precision? This is also a slide I borrowed from Ted Hansch. If you measure very, very accurately an atomic line, let's say the 1s 2s transition in hydrogen using two photon spectroscopy, what you can do is you can measure it, and a few years later, you can measure it again.

And now it becomes an interesting question if you have this precision. Will the result be the same as a function of time? If there were a small change, which there wasn't, you could only come to one conclusion, and this is fundamental common sense in nature, change is a function of time. So this precision of metrology is now being used to test the fundamental constants, again, constant as a function of time.

Precision is even more improving. The latest in the development in the spectroscopy of hydrogen is what is called the size of the proton atom. In your homework assignment, you are actually calculating what is the correction due to transition frequencies in hydrogen because you don't have a proton field of a point particle. The proton has a finite size. Or vice versa, if you have sufficient prediction, if you have sufficient accuracy of the measurement, you can determine the size of the proton from the measured transition frequencies.

This was done, and in 2010, there was a big surprise that the size of the proton determined from hydrogen spectroscopy did not agree with scattering measurements where you scatter electrons and protons to measure the proton size. This is still a puzzle. It's called the proton radius puzzle, and it is not clear what is causing it.

What happened is in 2010, there was a big improvement in measuring the size of

the proton, and this was done by replacing the electron with a muon, which is a heavy electron, but since the muon is so much heavier, the bore orbit of the muon, the negative particle going around the proton, it is much smaller. Therefore, the Lamb shift also, but also the correction due to the finite size of the proton, is much, much larger because there is much more overlay of the muonic wave function with a proton than for an electron.

So there was a huge improvement in the precision of the measurement of the proton size, and this has really led to what's called the proton radius puzzle. It's not sure if that is at the same level of the Lamb shift, which gave rise to fundamental new physics. Maybe this is the discovery of the new Lamb shift in 2010, and it changes our understanding of fundamental physics, but maybe it's something else. The answer is not [INAUDIBLE].

At least this is 2010. A few years of checking the theory and checking the experiment has not removed the discrepancy. Rather to the contrary. It has hardened that there is some discrepancy which needs to be resolved. So this was just a little shot of excursion, a little bit of summary of spectroscopy of hydrogen over 80 years, from precursors to the Lamb shift to the proton radius puzzle.

AUDIENCE: So I know the same group was using the same technique to measure deuterium, maybe helium

PROFESSOR: I think they want to do it but they haven't done it. That's what they plan to do.

AUDIENCE: That was my question.

PROFESSOR: So our next topic are now atoms in external magnetic fields. The first chapter is on fine structure and the lambda g factor. But maybe more colloquially, atoms in external fields means that we add one more vector to the mix.

In fine structure, we have orbital angular momentum and spin angular momentum, and we discussed how spin orbit coupling eventually couples L and S to J and so on, but now, we extend the game by one more vector, B, an external magnetic field. It really becomes a player in the game because you know that if you have spin orbit

coupling, we use the vector model that L and S couple and precess around the axis of J, the total angular momentum.

So the game we play when we couple angular momentum is that angular momentum couple, they precess around an axis that involves some quantum numbers and so on. But now, you can add a new quantization axis with a magnetic field, and then angular momenta will precess around the magnetic field. And there may be actually a conflict, that for strong magnetic field, the precession is different than in weak magnetic fields, and this is what we want to discuss now. So it is the game of L, S, and B.

So what we are adding with magnetic fields is we are adding one term to the Hamiltonian, which is the Zeeman term, where we have an external magnetic field and we couple with a magnetic moment. The one thing which makes it interesting when we discuss fine structure is the following, that we have actually two components to the angular momentum of the atom, which is the spin and the orbital angular momentum, and the two have different g factors.

So it's not that the magnetic moment is just proportional to the angular momentum. If the angular momentum comes from spin, it has a different weight than when the angular momentum comes from orbital angular momentum, and this is what we want to now understand. We want to determine what is the magnetic moment of the atom when the angular momentum has two different sources? What we will find is we will find that there is a lambda g factor which is sometimes 2, which is sometimes 1, or which is sometimes somewhere in between, depending how S and L are arranged with respect to each other.

So our Hamiltonian is the Hamiltonian for the atom. We know that we have fine structure, we discussed that, which couples L and S. And now have a magnetic moment due to the spin and due to the orbital angular momentum, and then couples to the external magnetic field.

In other words, if we had no fine structure coupling, if S and L would not be coupled, the answer would be very simple. S just couples to the magnetic field, gives the

same [INAUDIBLE], the g factor of 2, and L couples to the magnetic field with a g factor of 1. But now the two are coupled with respect to each other, and if you have L-S coupling, the projection of S and L on the z-axis is not a good quantum number anymore. So therefore, we have two different terms which are diagonal in two different bases, and that's what we want to discuss.

So the g factor of orbital angular momentum is 1. The g factor of the spin is 2, or if you want to include the leading correction here from QED, it's the fine structure constant over 2 pi. We did discuss that the fine structure can be related as the Zeeman energy of the spin in a magnetic field which is created by the electron due to its motion. Or, if you take the frame of the electron, the electron sees the proton orbiting around that creates a magnetic field, and this magnetic field couples to the state.

So therefore, we can associate fine structure with an internal magnetic field inside the atom. And this internal magnetic field is rather large. It's on the order of 1 Tesla. So therefore, for our discussion of the lambda g factor and fine structure in applied magnetic fields, we will assume that we are in the weak field limit where the fine structure term, the first term, is much larger than the Zeeman term.

Of course, if you use very strong magnets, you can go to the high field case, but I will discuss explicitly the transition from weak field to high field for hyperfine structure, and the phenomenon for fine structure is completely analogous. It just happens at much higher fields. So anyway, I will discuss the high field case and transition with the high field case with a much more elegant example of hyperfine structure that you can immediately apply to fine structure if you like.

If I want to solve the problem, calculate the lambda g factor, I could directly calculate just one matrix element and it would be done. So all I want to know is what is the Zeeman energy because Zeeman energy divided by the magnetic field is the magnetic moment. And since I assume that I'm in the weak field limit, I can simply use the quantum numbers S, L, and S and L coupled to J, and the magnetic quantum number is nJ. So by simply calculating this expectation value, I'm done

and I've solved the problem. However, I want to do the derivation using the vector model because it provides some additional insight.

So in the vector model, we have L and S coupled to J to the total angular momentum. And in the vector model, we assume that L and S rapidly precess around J. And therefore, the only thing which matters are the projections. Only the projections of L and S on the J-axis are important. So you can say if you have an rapid precession of L and S around J, the transverse components rapidly average out and [INAUDIBLE].

So therefore, our Zeeman Hamiltonian has to be rewritten in the following way. The Zeeman Hamiltonian was the magnetic moment times the external magnetic field with a minus sign. But what matters is the projection on the J direction, so we do the projection in this way. And also, in the end, what matters is, since the magnetic moments are aligned with J, it is now the scalar product of the magnetic field with J.

So in the vector model, we calculate the Zeeman energies in that way, but just to mention that if you don't like the vector model and the assumption of rapid precession, just take this matrix element. It's exactly the same. In other words, I give you the intuitive picture what is inside those matrix elements.

So let's evaluate that. Let me factor out the Bohr magneton. We have J squared, taking one of each bracket. And now assuming that the g factor of the spin is 2, the magnetic moment is the Bohr magneton times L-- the g factor of L is 1-- plus S, but the g factor of S is 2. So this is now the magnetic moment accounting for the two different g factors we projected on the J-axis.

And the second racket, B dot J becomes the value of the magnetic field. We assume the magnetic field points in the z direction, so therefore, it is the z component of the total angular momentum. Let me collect the simple terms. Now L plus 2S, because L plus S is J, can be written as J plus S.

So now we have the product of J with J, which gives us J squared, and then we need the product of S and J. And as usual, we can get an expression for that by

using the summation of angular momenta. If we square it on the right hand side, we have the scalar product of J and S, but we have now the scalar product of J and S expressed by L squared J squared. J squared plus S squared minus L squared divided by J squared.

Now, we're just one line away from the final result. Jz is a good quantum number. It's MJ, the projection of the total angular momentum on the z-axis. The bracket here is now the famous result for the lambda g factor. So we have J squared over J squared, which gives us 1.

And then I simply put in the quantum numbers for J squared, S squared, L squared, which is J times J plus 1 plus S times S plus 1 minus L times L plus 1, and we divide by 2 times J plus 1. So therefore, the Zeeman structure in a magnetic field is the Bohr magneton times the magnetic field times the angular momentum in the J direction, but then we multiply with the g factor.

These are now limiting cases. If we do not have spin, then the only ingredients, the only [INAUDIBLE] angular momentum is orbital angular momentum and we have a g factor of 1. So the lambda g factor simply becomes gL. In the case we don't have angular momentum, you can just evaluate this expression for L equals 0. You find indeed that the g factor is 2. But it can have different values. It depends on the atomic structure.

That's all I want to say about fine structure in a magnetic field. The next step is now hyperfine structure. We are adding one more vector to the game. So we add angular momentum of the nucleus, so now the game being played is not only L and S. We have I and B. It's a game of the four vectors and eventually how they precess around each other, and that gives rise to the structure of hyperfine levels in an external magnetic field.

We assume that L and S have coupled to J, so we have actually a coupling of J, I, but now we have an external quantization axis in Zeeman. It has to be due to the external magnetic field. And of course, in hyperfine structure, we discussed that I and J are no longer conserved angular momenta because they couple to a total

angular momentum, which is F.

So our Hamiltonian is the Hamiltonian without any kind of hyperfine and fine structure when we have the hyperfine coupling, which couples I and J with the product I dot J. And then we have an external magnetic field, which couples to the magnetic moment of the electron. This may be a smaller term, but you can easily carry with us. There's also a coupling with the magnetic moment of the nucleus.

So in this case, because the hyperfine structure is smaller than the fine structure, I want to discuss both the weak field and the strong field case. Because magnetic fields of a few hundred [INAUDIBLE] may actually take into the high field momentum. I want to discuss both the low field and the high field limit. The low field limit implies that the Zeeman energies are much smaller than the hyperfine splittings.

And then the way we describe the system is that J and I couple. So J, which is responsible for the magnetic moment, precesses around F, the total angular momentum, but then the total angular momentum precesses around the magnetic field. In other words, you assume that the coupling between J and I is so strong and they couple to F, the magnetic field is not breaking up the coupling between J and I. J and I together form F, and this hyperfine state, this magnetic moment, precesses around B.

This is sort of the picture. You have to get used to it. J precesses around F and F precesses around the external magnetic field, B0. But again, if you don't like the precession model, just calculate the quantum mechanical energy that will stabilize the Hamiltonian. The answer is identical.

The Zeeman Hamiltonian couples to the magnetic field, and we have two contributions to the magnetic moment, the electron of the nucleus. And in the weak field limit, we use a treatment which is almost completely analogous to the treatment we used when we derived the [INAUDIBLE] g factor. We can treat the same as Hamiltonian perturbation theory, and it's exactly analogous when we added a weak magnetic field to the fine structure.

In the vector model, we have the coupling of J and I to F. The relevant term in the Hamiltonian is we have the magnetic moment of the electron, which is proportionate to J and it couples to B. So this relevant term, this is fully analogous to what I did five or 10 minutes ago, has to be replaced in the presence of the nuclear angular momentum. We have to project everything on the axis of the total angular momentum, F.

Therefore, the Zeeman Hamiltonian had the contribution to the magnetic moment due to the electron and due to the nucleus. This is proportionate to J, but now we have to project it onto F. And similarly, the magnetic moment of the nucleus is proportionate to I, but what matters is the projection on F. And since I factored out the Bohr magneton, the magnetic moment of the nucleus is proportionate to the nuclear magneton. I have to account for the ratio. What matters now is the projection of F on B0.

So therefore, collecting all the terms, we have the Bohr magneton, which is setting the scale of the interaction. The last term is the magnetic field, but the projection of F onto the magnetic field gives us the mF quantum number, and all the rest is called the g factor of the hyperfine structure.

And the g factor of the hyperfine structure is-- let me just simplify and neglect the small contribution-- it's 1,000 times smaller-- of the nuclear magnetic moment, but if you want, you can easily include it. With this approximation, the g factor of the hyperfine structure is this. It's proportionate to the lambda g factor we just derived.

And then using exactly the same thing, you have J dot F. You can express it now by the quantum numbers of F squared, J squared, I squared. You find the final result, what are the g factors of the hyperfine levels, of the hyperfine states. So this is the hyperfine structure of atoms [INAUDIBLE] magnetic fields.

Let's immediately go to the high field limit. The high field limit means that the electronic Zeeman energy is much larger than the hyperfine coupling. And that means always, when we treat the problem, we first take care of the big contributions

to the Hamiltonian. We try to solve it, if possible, [INAUDIBLE]. And then the weaker term can be any perturbative. So now we are in the situation that the Zeeman coupling is the big term and the hyperfine coupling is the weaker term.

So in other words, what comes first now is the Zeeman energy, so we are not coupling the electronic angular momentum and the nuclear angular momentum to total angular momentum because this coupling is weak. We rather say that the electronic and the nuclear angular momentum align with the magnetic field. We quantize along the direction of the magnetic field, and then later we add the hyperfine coupling in a determinative way.

So B0 now quantizes J along the direction of the magnetic field, and therefore we use, as a good quantum number, J, the projection of J on the external magnetic field axis. So this takes care of J. J and mJ are good quantum numbers.

What about I? Does the nuclear angular momentum and the nuclear magnetic moment strongly couple to the magnetic field? Well, the answer is yes, but the argument is a little bit more subtle. The direct coupling of the magnetic moment of the nucleus with the magnetic field may be smaller than the hyperfine interaction. So then you would say, the nuclear angular momentum should not couple to the magnetic field. It should first be coupled to the hyperfine interaction.

The hyperfine interaction is I dot J. However, J, which couples strongly to the magnetic field because it couples to the Bohr magneton, has already been coupled to the magnetic field. So therefore, the hyperfine interaction, which is I dot J, is now modified because J couples to the magnetic field, which means we have to project it onto the magnetic field axis. So therefore, the nucleus now experiences an electronic magnetic moment or electronic angular momentum, which has already been coupled to the z-axis. And therefore, the hyperfine interaction is also coupling the nuclear angular momentum to the z-axis.

So therefore, the result is that it is now this indirect coupling. You couple the electron angular momentum to the z-axis and the electron angular momentum couples the nuclear angular momentum to the z-axis. So now this quantizes the

nuclear angular momentum along the z-axis, which means that n sub I becomes a good quantum number.

Anyway, maybe the result is even simpler than the explanation. Our Zeeman Hamiltonian now simply means that we have an external magnetic field and you the electron couples to the magnetic field, so what matters is the projection, mJ. The same happens for the nuclear magnetic moment, and now we have to add the hyperfine interaction, which was originally I dot J, but since I and J are projected on the z-axis, what is really left over are only mI and mJ.

I could have gotten this expression immediately by just telling you, J and I no longer couple to F. This is destroyed by a strong magnetic field, and the good quantum numbers are J and I and their projection, mJ and mI. And then just writing down the expectation value of the Hamiltonian in these spaces would have immediately given me this result. I wanted to give you the more mechanistic explanation what's going on inside the atom and what leads to this result.

I have discussed for you the two limiting cases, the weak field and the strong field case, but you can solve it also for intermediate fields. You simply have to do an exact diagonalization of the Hamiltonian, which involves the hyperfine coupling. And the hyperfine coupling, if you want, can be diagonalized as eigenfunctions where the quantums numbers are J, I coupled to F, and the projection of F, the magnetic quantum number is mF.

But now we have the Zeeman Hamiltonian, where everything is projected on the z-axis, so we have mJ and mI. And the Zeeman term can be diagonalized in a different basis, which is the basis of J, I, mJ, and mI. So I've shown you the weak field limit, where we simply assume those quantum numbers and calculate this term determinatively, and I've shown you the high field limit, where we used those quantum numbers and calculated this field determinatively.

But in general, you just have to write down the matrix element of this Hamiltonian in whatever basis you choose. You can use the weak field basis. This term is diagonal. This is off diagonal. Or you can use the strong field basis where this is the diagonal

and this is off diagonal and simply diagonalize your Hamiltonian. Find the wave function, find the I energies. And since, for cases where S equals 1/2, it's only a two by two matrix which has to be diagonalized, you can do it analytically, and this leads to the famous [INAUDIBLE] formula.

So the solution is analytic for J equals 1/2, and it's a beautiful example which you should solve in your homework assignment. Let me just sketch the solution. When you go from the weak field to the strong field limit, the z component of the total angular momentum in one case is mF. In the other case, it is mI plus mJ. So when you go from one limit to the next, you connect only states where mI plus mJ equals mF. So the structure of the general solution can be explained by repulsion and anti-crossings of states with the same [INAUDIBLE] number.

So let me show you the weak field, the strong field limit, and do a graphic interpolation. What I've chosen as an example is the case of a 2 duplet S 1/2 ground state and a nuclear angular momentum of 3/2. Examples for that is sodium and rubidium 87.

This is magnetic field. At weak magnetic field or zero magnetic field, the spin 1/2 couples to the nucleus in 3/2, and that gives to hyperfine states F equals 1 and F equals 2. The splitting is given by the hyperfine constant, a, and the hyperfine interaction is A times H times I times S. The structure is such that the center of mass of the energy levels is preserved, so therefore, one state is moved up by 3/4 a. The other state is moved down by 5/4. And since F equals 1 has three components, F equals 2 has five components, the center of mass is preserved.

We have calculated the g factor for those states, and the g factor tells us what is the structure in weak magnetic fields. So this is the weak magnetic field solution. At high magnetic field, you know what you have in high magnetic field. You have a single electron which can spin up and spin down. So if you have an electron which is spin up and spin down, it pretty much is linear Zeeman shift for spin down and linear Zeeman shift for spin up.

This is sort of what we expect. So what will happen is that the energy levels will

evolve like this. So in other words, we have eight levels. We have the structure at weak magnetic fields. At high magnetic fields, of course, we also have eight levels, but they pretty much group into spin up of the electron, spin down of the electron. And then there is a smaller hyperfine structure on top of it because now the nuclear spin can have various orientations. And I equals 3/2 state has four orientations, so therefore, electron spin up and electron spin down will obtain four sub-levels.

And if you ask how did I connected, I've connected the quantum numbers as such what is here, the states are labeled by mI and mS, and here, they're labeled by mF, but mF equals mI plus mS. This is how you correlate the states in the high field case to the states in the low field case. I'm running out of time, but here, we have mJ equals 1/2. Here we have the electron spin minus 1/2. These four levels are now four different quantum numbers for the nuclear angular momentum, which are minus 3/2, minus 1/2, plus 1/2, and plus 3/2.

And this is what I meant by avoided crossing. At some point, I think, draw it yourself, put the quantum numbers on it, and you'll learn a lot by doing it. What you realize also when you solve the Hamiltonian that this structure can be explained the following way. You will always find you have some states which are stretched, where there is only one state which has the maximum angular momentum inside all the stretched states.

And then the other states, you always need to find two states which have the same total mF, and those two states avoid each other. You can say, just pointing on two states, those two states, let's just assume they have the same mF. They undergo an avoided crossing, and that's exactly what you get out of the diagonalization of the two by two matrix. So this whole diagram can be understood by you have stretched states which form a one by one matrix. There is no recoupling taking place.

And then you have three pairs of states which form two by two matrix, and in each pair, if you would now focus on it, you really see the avoided crossing typical for a two by two matrix. Any questions about that? The next thing would be to go through

some bigger questions and review atomic structure, including external magnetic fields. But we'll do that at the beginning of the next class.