PROFESSOR: OK, let's start. So we've been looking at the xy model in two dimensions. It's a collection of units spins located on a site of potentially a lattice in two dimensions. Since they are unit vectors, each one of them is characterized by an angle theta i.

And the partition function would be obtained by integrating over all of the angles. And the weight that we said had the form k e cosine of theta i minus theta j. So there's a coupling that corresponds to the dot per dot of neighboring spins, which can be written as this cosine form.

If we go to low temperatures where k is large, then this was roughly an integral over all configurations of the continuous field, theta of x, with a weight that is the appropriate choice of the lattice spacing, the same k integral of gradient of theta squared. Now if we look at this weight and ask what is happening as a function of changing temperature or the inverse of this parameter so that we are close to zero temperature, if we just work with this Gaussian, the conclusion would be that if we look at the correlation between two spins that are located at a distance r from each other, just from this Gaussian weight, we found that these correlations decay as something like 1/r, or maybe we put some kind for a lattice spacing, to the power of an exponent theta that was related to this k. So the conclusion was that if that is the appropriate weight, we have power-law decay correlations. There is no long range order for correlations decay very weakly.

On the other hand, if we start with the full cosine and don't do this low temperature expansion, just go and do the typical high temperature expansion, from the high temperature expansion we conclude that these collections decay exponentially, which are two totally different forms. And presumably, there should be some kind of a critical value of k or temperature that separates a low temperature formed with

power-law decay and a high temperature formed with exponential decay of correlations.

Now there was no sign of such a kc when we tried to do a low temperature expansion like the non-linear sigma model in this particular case of the xy model that corresponds to n equals to 2. Again, we said that the reason for that is that the only high order terms that I can write down in this theory close to low temperature gradient of theta to the fourth, sixth, et cetera are explicitly irrelevant. And unlike n equals 3, et cetera, it cannot cause any change. So as far as that theory was concerned, all of these corresponded to being fixed points.

Then we said that there is a twist that is not taken into account when I make that transformation in the first line, as pointed out by Kosterlitz and Thouless. What you have are topological defects that are left out. And an example of such a defect would be a configuration of spins that are kind of radiating out from some particular point, such that when I complete a circuit going around, the value of spin changes by 2 pi for the gradient at the distance r, gradient of the change in angle should fall off as 1 over r.

So if we put one of these defects and calculate what the partition function is for one defect, z of one defect, we would say, OK, what I have to do is to calculate the energy cost of this distortion. If I use that theory that I have over there, I have k/2. I have integral over all space which, because of the symmetry here, I can write as 2 pi r to the r.

And then I have the gradient squared. And as I said, the gradient for a topological defect such as this goes as 1/r. It's square goes like 1 over r squared. So this is an integral that is logarithmically divergent. It's a an integral of 1/r. It's logarithmically divergent both at large distances-- let's say of the order of the size of the system-- and has difficulty at short distances. But at short distances, we know that there is some lattice structure, and this approximation will break down when I get to some order of some multiple lattice spacing-- so let's call that a-- and that the additional energy that comes from all of the interactions that are smaller than this core value

of a I have to separately calculate, and I'm going to call them beta epsilon that depends on the distance a that I choose for the core.

This is just a energy term that goes into the Boltzmann factor. But this defect can be placed any place in a lattice that has size L up to this factor of 1/a that I call the size of the core. So the number of places goes like L/a squared, the area that I'm looking at. And so we see that this expression actually goes like L/a.

I have the power 2 here. And from this logarithmic interaction, I will get here power of 2 pi k log L/a, which I can absorb into this. And I define the exponential of the core energy to be some parameter y. That clearly depends on the choice of what I call my core.

So if I look at just that expression by itself, I would say that there is the chance to see one defect or no defects as L/a becomes large in the thermodynamic limit depending on whether this exponent is positive or negative. So we are kind of-- oops, I forgot the factor of 2 here. It's 1 over pi k.

We are led from this expression that something interesting should happen at some kc for one defect, which is let's call it the inverse of the kc. That is like a temperature, which is pi/2. So maybe around pi/2 here there should be something interesting that happens.

In fact, if the theory was independent vertices, I would predict there would be no vertices up to here, and then there would be whole bunches vertices that would be appearing later on. But the point is that these are not vertices that don't interact with each other. There are interactions between them.

And if I look at a situation with many vertices, what I need to do is to calculate the partition function for the defects, for the vertices that has a resemblance to the Coulomb system, so we call it z sub q. And this z sub q is obtained by summing over all numbers of these defects that can appear in the system. So let's do a sum over n, starting from 0 to however many. And if I have a situation in which there are n vertices in the system, I clearly have to pay a cost of y per core of each one of

them. So there's a factor of y raised to the power of n.

These vertices then can be placed anywhere on the lattice, in the same way that I had this factor of L/a for a single vertex. I will have ability to put each one of these vertices at some point on the lattice. And then we found that when you do the calculation, the distortions that are caused by the independent vertices clearly add up on top of each other. And when we add it up and superimpose the gradient of thetas that correspond to the different vertices and calculated the integral of gradient of theta squared, what we found was that there was an interaction between them that I could write as 4 pi squared k sum over distinct pairs, i less than j, qi qj. The Coulomb interaction [? it fits inside ?] xi and xj.

And actually, the form of this came about as follows. Those That basically the charges of these topological defects are multiples of 2 pi, but these qi's I have written are minus plus 1. So the two pis are absorbed in the charges. The actual charges being 2 pi is what makes this 4 pi squared. k is the [INAUDIBLE] of interaction.

Furthermore, we have to require the system to be overall neutral because, otherwise, there would be a large energy for creating the monopole in a large system. And again, just as a matter of notation, our z of x has a 1 over 2 pi itself log of the displacement in units of this a because we can't allow these things to come very close to each other. So our task was to calculate properties encoded in this partition function, which is, in some sense, a grand canonical system of charges that can appear and disappear.

And our expectation is that at the low temperatures, essentially all I have are a few dipoles that are kind of small. As I go to higher temperature, the two monopoles making the dipole can fluctuate and go further from each other. And eventually at some point, they will be all mixed up together and the picture should be regarded as a mixture of plus and minus charges in the plasma. Yes?

AUDIENCE: [INAUDIBLE] if we have an external field, would this also be fixed [INAUDIBLE], like and edge or something external?

PROFESSOR: It is very hard to imagine what that external field has to be in the language of the xy model, because what you can do is you can put a field that, let's say, rotates the spins on one side-- let's say to point down-- spins on the other side to point up. But then what happens is that the angles would adjust themselves so that at 0 temperature, you would have a configuration that would go from plus to minus, and all topological charges would be on top of that base configuration. So that kind of field certainly does not have any effect that I can ascribe over here.

If I change my picture completely and say forget about the xy model, think about this as a system of point charges, then I can certainly, like I did last time, put an electric field on the system and see what happens. Yes?

AUDIENCE: [INAUDIBLE] saying that you can create even bigger defects, where q would be [INAUDIBLE] plus minus 1, plus minus bigger integer? But that's [? discounted ?] as high [INAUDIBLE] effect.

PROFESSOR: Yes. So intrinsically, we could go beyond that. We would have a fugacity for a creation of cores of singular charge, another for cores of double charge, et cetera. You expect those y's or double charges to be much larger because the configuration is going to be more difficult at the core. And in some sense, you can imagine that we are including something similar to that because we can create two single charges that are closing off to each other. Yes?

AUDIENCE: So why is a Coulomb state [INAUDIBLE]? Is that high order?

PROFESSOR: Well, you would expect that if y is a small parameter, you would like to create as few things as possible. The reason you create any is because you have entropy gain. So I would say energetically, even creating a pair is unfavorable.

But the pair has lots of places that it can go, so because of the gain in entropy, it is willing to accept that. If I create a quadrupole, you say, well, I break the quadrupole into two dipoles and then I have much more entropy. So that's why it's not-- we can have that term, but it is going to have much less weight.

So I won't repeat the calculation, but last time we indeed asked what happens if we have, let's say, some kind of an electric field. And because of the presence of the electric field, dipoles are going to be aligned. And the effect of that is to reduce the effective strength of all kinds of Coulomb interactions. We found that the effective strength was reduced by from k by an amount that was related to the likelihood of creating the dipole of size r.

And that was clearly proportional to y squared into the minus from there, 4 pi squared q log of r/a. That's the co-ability we create a dipole of this size. And then I have to, in principle, integrate over all dipole sizes. But this writing this as an orientationally independent result is not correct because in the presence of an electric field, you have more likelihood to be oriented in one direction as opposed to the other direction. So this factor of e to the cosine theta, et cetera, that we expanded, first of all, gave us an average of cosine of theta squared. So there was a factor of 1/2 here. Rather than full rotation, I was doing an average of cosine squared to be that factor.

Expanding that actually gave me a factor of 4 pi squared k because of the Coulomb term that I had up there. And then I had essentially the polarizability of one of these objects that goes like r squared. Again, coming from expanding this factor that we had in the exponents.

Actually, I calculated everything in units of A, so I should really do this. And this was correct to order of y squared. And in principle, one can imagine that there are configurations of four charges, quadrupole, like things, et cetera, that further modify this.

And that this is a result that I have to-- oops, this was 1 minus. It was an overall factor of k. This was the correction term that we calculated. And then the size of these dipoles, we have to integrate from A to the size of the system or, if you like, infinity.

And although we were attempting to make an expansion in powers of y, what we see is that because this is giving me a factor of k/r, the r has to be integrated

against these three factors of rdr. Whether or not this integral is dominated by its upper cut-off, and hence divergent, depends on value of k that is related to the same divergence that we have for a single vertex. So this perturbation theory is, in principle, not valid no matter how much y I try to make small as one long as my k inverse is greater than pi/2.

So what we decided to do was not to do this entire integration that gives us infinity, but rather to recast this as a re-normalization group in which core size is changed from a to beta. Now one way to see the effect of it-- last time, I did this slightly differently-- is to ensure that the result for the partition function of one charge is unmodified. If I simply do this change, the weight should not change for one defect. And so clearly, you can see that there's a change in power of beta that I would that I need to compensate by changing the core energy by a factor of b to the power of 2 minus pi k.

So the statement that I have for z 1, in order for z 1 to be left invariant, I have to rescale the core energies by this factor. And then over here, I essentially just integrate up to a factor of da. Just get rid of those interactions. And so this becomes minus 4 pi q k squared y squared-- as we're looking at dipole contribution-- integral from a to ba of dr r cubed divided a to the fourth. a/r to the power of 2 pi k.

It means that I probably made a mistake somewhere. Yeah, this has a 2 pi. I forgot the 2 pi from the definition of the log.

So these are the recursion relations. So basically, this same results at large scale for the Coulomb gas can be obtained either by the theory that is parametrized by y and original k, or after going through this removal of short distance degrees of freedom by theory in which y is modified by this factor and k is modified by this factor. And as usual, we can change these recursion relations into flow equations by choosing the value of b that is very close to 1, and then essentially converting these things to y evaluated at a slightly larger than 1, and from that, constructing the y by dl. And y by dl simply becomes 2 minus pi k times y.

And I can do the same thing here. This is k plus dk dl. The k part cancels. And what

I will get is that dk by dl is minus 4 pi cubed k squared y squared. And actually, all I need to do is evaluate this on the shell where r equals to a, and you can see that the integral essentially gives me 1.

It gives you a delta l, basically goes over here. So these are order of y squared. Actually, it is kind of better to cast results rather in terms of k, in terms of k inverse, which is kind of like a temperature variable.

And then what we get is that d by dl of k inverse essentially is going to be minus 1 over k squared by dl of k squared. So the minus k squared cancels, and it simply becomes 4 pi q y squared, order of y to the fourth. And divide by dl, it's actually 2 minus pi k y plus O of y.

So these are the equations that describe the changing parameters under rescaling for this Coulomb gas. And so we can plot them. Essentially, we have two paramters-- y, and we have k inverse. And what we see is that k inverse, its change is always positive.

So the flow should always be to the right. y, whether y decreases or decreases depends on whether I am above or below this critical value of 2/pi that we keep encountering. And in particular, what we will find is that there is a trajectory that goes into this point.

And if you are to the left of that trajectory, y is getting smaller, k inverse is getting larger. And so you go like this. Eventually, you land on a point down here where y has gone to 0. And if y has gone to 0, then k inverse does not change.

So you have a structure where you have a line of fixed points so any point over here is a fixed point, but it is also a stable fixed point. It is true that points that are over here, if you are exactly at y equals to 0 are fixed points. But as soon as you have a little bit of y, then they start flowing away. And essentially, the general pattern of flow is something like this.

So I go back to my original xy model, and I'm at some value at low temperatures-- means that I'm down here-- but presumably, there's a finite cost for creating the

core, so I may be over here. And when I go to slightly higher temperature, k inverse becomes larger. But the core energy typically becomes smaller also at lower temperatures because everything is scaled by scale by 1/kt. So as I go to higher and higher temperatures, my xy model presumably goes through some trajectory.

The trajectory of changing the xy model as temperature is modified has nothing to do with [? rg ?]. So basically is xy model on increasing T. And what is happening in the xy model of increasing T is that at low temperatures, I'm at some point here, which if I look at larger and larger scales, I find that eventually I go to a place where the effective whole cost for creating vertices is so large that they are not created at all.

So then I'm back to that theory that has no vertices and simply gradient squared, and I expect that correlations will be given by this power law type of form. However, at some point, I am in this region. And when I'm in this region, I find that maybe even initially the core energy goes down or y goes down. But eventually, I end up going to a regime where both y is large and effective temperature-- the inverse-- are large.

So essentially anywhere here eventually at large scales, I will see that I will be creating vertices pretty much at ease and at sufficiently long, large scale. My picture should be that of a plasma in which the [? plas ?] plus and minus charges are moving around. And so then there is this transition line that separates the two regimes.

So let's find the behavior of that. And clearly, what I need to do is to focus in the vicinity of this fixed point that controls the transition. That is, anything that undergoes the transition eventually comes and flows to the vicinity of this point.

So what we can do is we can construct, if you like, a two dimensional blow-up of that. And what I'm going to do is to introduce a variable of x, which is k inverse minus 2/pi. Essentially, how far I have gone from this in this direction-- y, I can use as y [INAUDIBLE]. And so what we see is that my k inverse is 2/pi, my critical value.

AUDIENCE: I think that should be a pi/2.

PROFESSOR: k inverse this is pi/2. Thank you. Which means that this has to be pi/2. This has to be pi/2. And this I can write as pi/2, 1 plus 2x/pi.

So that to lowest order in x, k is 2/pi. The inverse of this factor, which is 1 minus 2x/pi plus order of x squared. I'm expanding for small x. I put that value in here and I find that my dy by dl is now 2 minus pi times what I have over there.

So it is pi times 2/pi-- so it becomes 2-- plus 4/pi x. So essentially, I have minus 4/pi squared. I multiple by pi. It becomes plus 4/pi x. Multiply by y. So this is simply a 4/pi xy.

Now the point is that typically we are used to expanding in the vicinity of that important fixed point. And all the cases that we had seen so far, once we did that expression, we ended up with a linear behavior-- divide by dl plus something times y. Here we see that the vicinity of this point is clearly a quadratic type of behavior.

And this quadratic behavior leads to some unusual and interesting critical behavior that we are going to explore. So let's stick with this a little bit longer. We can see that if I look at d by dl of y squared, it is going to be 2y divided by dl, so I have to multiply this by 2y, so I will get 8/pi pi xy squared.

Why did I do that? It's because you can recognize his xy squared shortly. Let's go and do the x by dl.

The x by dl is simply dk inverse by dl, so that is 4 pi cubed y squared. And now you can see that if I do d by dl of x squared, I will have 2x dx by dl, so I will have 8 pi cubed xy squared. So now we can recognize that these two quantities up to some factor of pi to the fourth are really the same thing.

So from here, we conclude that d by dl of x squared minus pi to the fourth y squared. Essentially once I do that, I will get 0. So as I go along these trajectories, x and y are changing, but the combination x squared minus pi to the fourth y squared is not changing.

So all of the trajectories that I have drawn-- at least sufficiently close at this point around which I am expanding-- correspond to lines that are x squared minus pi to the fourth y squared is some constant I'll call c. And that constant must meet whatever you started with. So if I call the trajectory here to be the combination x0 by x0-- your original values-- I can figure out what my x0 to the fourth minus pi the fourth x0 to the fourth is. And that's going to be staying constant along the entire trajectory.

So these trajectories are, in fact, portions of a hyperbole. And this is the equation that you would have for a hyperbola in xy. Now clearly there are two types of hyperbole-- the ones that go like this and the ones that go like that. In fact, this one and this one are pretty much the same thing.

And what distinguishes this pattern versus that pattern is whether this constant c is positive or negative because you can see that out here, ultimately you end up at the point where y has gone to 0. So depending on x positive or negative, it doesn't matter. This combination will be positive.

So throughout here, what I have is that c is positive. Whereas what I have up here is c that is negative. Presumably, there is other trajectory here and down here, c is again positive. And again, if you want to ensure y over here c is negative, because over here you can see you crossed the line where x is 0, but you have some value of y.

So if I were to blow up that region both as a function of x and y, well, first of all, I will have a particular set of trajectories-- the ones that end up at this important fix point, which correspond clearly to c cos to 0. So that c cos to 0 will give me two straight lines. So presumably there is this straight line, and then there is another straight line goes out there.

And then I have this bunch of trajectories that are these hyperbole that end up over here. I can have hyperboles that will be going out. And then I will have hyperboles that are like this. This are all in the high temperature phase, so let's [INAUDIBLE] like this.

So one thing that you immediately see is that the location of the transition that is given by this critical line when c equals to 0. So statement number one that we can get is that the transition line corresponds to c equals to 0. So solving for x as a function of y, I will get that x critical is either minus or plus pi squared y. Clearly from the figure, the solution that I want is the one that corresponds to minus.

My x was k inverse minus pi/2. This is kc inverse minus pi/2. And so what I see is that kc inverse-- the correct transition temperature-- is, in fact, lower than the value of pi/2 that we had deduced, assuming that there is only a single vertex in the entire system by an amount that to lowest order is related to the core energy or core fugacity. And presumably, there are higher order terms that I haven't calculated.

So this number that we have calculated by looking at a single defect, we can see that in the presence of multiple defects, starts to get lower. And this is precisely correct in the limit where y is a small quantity. Now that's a transition line. We can look to the left or to the right.

Let us just look at the low temperature phase. So for the low T phase, we expect c to be negative. And I can, for example, make that explicit by writing it as-- OK, so c was x0 squared minus pi to the fourth y0 squared-- what the starting parameters of the system dictate.

If you are under low temperature phase such that c is negative, it means that you are at temperatures that are smaller than Tc. So let's see, T minus Tc, let's write it as Tc minus T-- would be positive. But this has to be negative, so let me just introduce some parameter b.

It's not the same b as here. Just some coefficient that has to be squared. So I know that as I hit Tc, this c goes to 0. If I'm slightly away from Tc along the trajectory that I have indicated over here, right here I'm 0, so right I'm slightly negative. And there's no reason why the value that I calculated from x0 squared minus pi to the fourth y0 squared should not be an analytical function.

So I have expanded that analytical function, knowing that that Tc is equal to 0. There will be higher order terms for sure, but this is the lowest order term that I would have in that expansion. And as we said, this is preserved all along the trajectory that ends on this point.

So along that trajectory, this is the same as x squared minus pi to the fourth y squared, which means that I can write y squared to be 1 over pi to the fourth. x squared plus b squared Tc minus T. So if I want to solve for this curve, that's what I will have for some value of this quantity.

And what I do is I look at what that implies for the xy dl. The xy dl is 4 pi cubed y squared. I substitute the y squared that I have over there. I will get 4/pi times x squared plus b squared Tc minus T.

So under rescaling, this tells me what is happening to x. And in particular, what I can do is to integrate this equation. I have dx divided by x squared plus b squared Tc minus T is 4/pi dl-- just rearranging this differential equation. And this I can certainly integrate out to l. This you should recognize as the differential form of the inverse tangent up to a factor of 1 over b square root of Tc minus T.

So I integrate this. And on the other side, I have 4/pi l. So eventually, I know that-- that's what I wanted to do? I needed to do this later on, but we'll use it later on.

What I needed to get is what is the eventual fate of this differential equation. Eventually, we see that this differential equation arrives at the point that I will call x infinity. When it arrives at x infinity, this is 0 and y is 0, so I immediately know that x infinity-- I didn't need to do any of that calculation-- this expression has to be 0, is minus the square root of Tc minus T. So let me figure out what I did with the signs that is incorrect.

AUDIENCE: [INAUDIBLE] temperature [INAUDIBLE] positive [INAUDIBLE].

PROFESSOR: In the low temperature phase, I have indeed stated that c has to be positive, which means that this coefficient better be positive, which means that I would have a minus sign here. And then x would be b times Tc minus T. Right, so this would be

plus or minus.

The plus solution is somewhere out here, which I'm not interested. The solution that I'm interested corresponds to this value. You say, well, what is important about that?

You see that various properties of this low temperature phase are characterized by this power law as opposed to exponential behavior. The power law is determined by the value of k where the description in terms of this gradient squared theory is correct. Now out here, the description is not correct because I still have the topological difference.

But if I look at sufficiently large distances, I see that the topology defects have disappeared. But by the time the topological defects have disappeared, I don't have the original value of k. I have a slightly different value of k.

So presumably, the properties are going to be described by what value of this k inverse is of large behaviors. And so what I expect is that the effective behavior of this k inverse-- actually, the effective behavior of k-- as a function of whatever the temperature of the system is. We expect that in the original xy model, or any system that is described by this behavior, there is a critical temperature, Tc, such that at higher temperatures, correlations are decaying exponentially. So essentially, the effective value of k has gone to 0. There is no stiffness parameter.

So basically at high temperatures, you should be over here. What I see is that the effective value of k, however, other is meaningful all the way to the inverse of 2/pi. So there is a value here at 2/pi which corresponds to the largest temperature or the smallest k that is acceptable. Now what I see is that on approaching the transition, the value of k-- I have it up there-- is 2/pi. This limiting value that we have over here.

And then there is a correction that is 4 over pi squared x. And presumably here, I have to put the x in infinity. And what I have for the x infinity is something like that, so I will get 2/pi plus 4 e over pi squared square root of Tc minus T.

So the prediction is that the effective value of k comes to its limiting value of 2/pi with a square root singularity. So we can replace the theory that describes anything

that is in this universality class in the temperature phase by an effective value of k. If we then ask how does that effective value of k change as a function of temperature, the prediction is that, well, at very low temperature, it's presumably inversely related to temperature. It will come down, but [INAUDIBLE] it will change its behavior, come with a square root singularity to a number that is 2/pi, and then jump to 0.

Now you are justified in saying, well, this is all very obscure. Is there any way to see this? And the answer is that people have experimentally verified this, and I'll tell you how.

So a system that belongs to this universality class and we've mentioned all the way in the class is the superfluid. We've said that the superfluid transition is characterized by a quantum order parameter that applies a magnitude, but then it has a phase theta. And roughly, we would say that the phase theta should be described by this kind of theory at low temperatures.

So if we want basically a two dimensional system, what we need to do is to look at the superfluid field. And this is something that Bishop and Reppy did in 1978 where they constructed the analog of the Andronikashvili experiment that we mentioned in 8333, applied it to the field. So let me remind you what the Andronikashvili experiment was.

Basically, you will have a torsional oscillator. This torsional oscillator was connected to a vat that had helium in it. So basically, this thing was oscillating, and the frequency of oscillations was related to some kind of a effective torsion of constant k divided by some huge mass which is contained within the cylinder. So basically you can probe classically-- you would say there's some kind of a density here. You can calculate what the mass is if you know what this is, you know what the omega is.

Now what he noticed was that if this thing was filled with liquid helium and you went below Tc of helium, then suddenly this frequency changed. And the reason was that the mass that was rotating along with this whole thing was changed because the part that was superfluid was sitting still, and the normal part was the part that was oscillating.

So the mass that was oscillating was reduced, frequency would go up. And from the change in frequency, he could figure out the change in the density of the part that was oscillating, and hence calculate what the density of the normal part was. So what Bishop and Reppy did was to make this two dimensional. How did they make it two dimensional?

Rather than having a container of helium, what they did was they made, if you like, some kind of a toilet paper. They call it a jelly roll my Mylar. So it was Mylar that was wrapped in a cylinder. And then the helium was absorbed between the surfaces of Mylar. So effectively, it was a two dimensional system in this very setup.

So for that two dimensional system, they-- again, with the same thing-- they measure the change in frequency. They found that if they go to low enough temperatures, suddenly there is a change in frequency. Of course, the temperature that they were seeing in this case was something like 1 degrees Kelvin or a fraction of 1 degree Kelvin, whereas when you have the full superfluid, it's 2.8 degrees Kelvin clearly because of that dimensionality the critical temperature changes. But you would say that's not particularly the inverse.

So they could measure the change in frequency and relate the change in frequency to the density that became superfluid. Now how does the superfluid density tell us anything about this curve? Well, the answer is that everything is going to be weighted by something like e the minus beta times some energy.

The one part of the energy that is associated with oscillations is certainly the kinetic energy. So let's see what we would write down for beta times the kinetic energy of superfluid or superfluid film. What I have to do is beta will give me 1/kt. The kinetic energy is obtained by integrating mass times velocity squared, or density integrated against velocity. It's a two dimensional film, so we sort of integrate as we go along the film.

The superfluid velocity can be related to the mass of helium h bar and the gradient of this phase of the superconducting order parameter. So you can, for example,

write your weight function as sidebar into the i theta of x, calculate what the current is using the usual formula of h bar over m psi star [? grat ?] psi [? minus ?] psi [? grat ?] psi star, and you would see that effective mass is something like this.

So this is going to give me rho over kt h bar over n helium 4 squared integral gradient of theta squared, which you can see is identical to the very first line that I wrote down for you. And we can see that k can be interpreted as rho-- kt h bar over n squared. So all of these quantities-- h bar m, you know. T is the temperature that you're measuring. Rho you get through the change in this frequency.

And so then they can plot what [INAUDIBLE] rho is as a function of temperature. And we see that it's very much related to k. And indeed, they find that the rho that they measure has some kind of behavior such as this.

And then they go and change their Mylar, make the films thicker or whatever. They find that the transition temperature changes so that a different type of film would show a behavior such as this. And thicker films will have a higher critical temperature. They do it for a number of film thicknesses, and they got things' behavior such as this, and found that this behavior followed this gray line, which is exactly what is predicted from here. It was predicted that rho c over Tc should kb m over h bar squared times what the critical value of k is that we've calculated to be 2/pi

So they could precisely check these 2/pi that we've calculated. They could more or less see this square root approach to this singularity. I'm not sure the data at that point were good enough so that they could say this exponent is precisely 1/2.

So this was for the low temperature phase. What can I say about the high temperature phase? So in the high temperature phase is where my c is negative.

So there I can write x0 squared minus pi to the fourth y0 squared as being a negative number, which I will write as minus b squared T minus Tc. So I have now T that is greater that Tc, multiply with some constant, and I get this. And this is the same all along the trajectory. So as I go further and x and y change, they will

change in a manner that is consistent with this, which implies that as x changes with l and y changes with l, the two of them will be related by y squared is being x squared plus b squared T minus Tc divided by pi to the fourth.

So this is where I don't really see the endpoint of the trajectory. I just want to see how the trajectory is behaving. So I go back to this equation, dx by dl is 4 pi cube y squared. Substitute that y squared, I will get 4/pi 1 over x squared plus b squared T minus Tc.

And then I rearrange this in a form I can see how to integrate. dx x squared plus b squared T minus Tc is 4/pi dl. I integrate the left-hand side. And as I already jumped ahead, it is the inverse tangent of x divided by b square root of T minus Tc is 4/pi times l.

So what do I want to do with this expression? So what I want to do is to see the trajectories that just cross to the high temperature side. So I start with a point that is just slightly to the right of this transition line.

Presumably, what is happening is that I will follow the transition trajectory for a long while, then I will start to head out, which means that for this trajectory, if I look at the system over larger and larger scales, initially I find that it becomes harder and harder to create these topological defects. The core energy for them becomes large. The fugacity for them becomes small.

But ultimately I manage to break that, and I go to a regime where it becomes easier and easier to create these topological defects. And presumably at some point out here, everything that I have said I have to throw out because I'm making an expansion, assuming that y is small, x is small, et cetera. So presumably, as I integrate, I come to a point where I say, OK, differential equations break down.

But my intuition tells me that I have reached the regime where I can create pretty much plus and minus charges at ease. So I would say that once I have reached that region where x and y have managed to escape the region where they are small, they have become of the order of 1. Maybe you can put them 1/3, 1/4-- it doesn't

matter.

Once they have become something that is not infinitesimal, then I can create these charges more or less at will. I will have a system where I have lots of charges that can be created at ease. And my intuition tells me that in that system, I shall have this kind of decay-- exponential decay.

So how far did I have to go in order to reach that value of l? I have to go to a correlation length for a size that is larger than 1 what I started with by a factor of e to the l where the value of x became something that is of the order of 1. And actually, you can see from here that if I'm very close to the transition, it doesn't matter whether I choose here to be 1/10, 1/2, even 1/100. As long as I'm close enough to Tc, I'm dividing something by something that is close to 0.

And this is tan inverse of a large number. And tan inverse of a large number is tan inverse of 90 degrees. So essentially, I go to some value of x where I can approximate this by pi over 2.

And you can see that that is really insensitive to what I choose to be my x's as long as I'm sufficiently close to the critical point. So you can see that once I have done that, I have figured out what my l star is, if you like. And if I substituted that over there, I will get a behavior that is about pi/4 times pi/2 times 1 over the square root of T minus Tc.

Now these coefficients out front are not that important. What you see is that, indeed, we get a correlation length that, as we approach Tc, diverges, but it is not that at all of any of the forms that we had seen before. So typically, we wrote that the correlation lens diverges T minus Tc to some exponent minus mu.

This is not that type of divergence. It's a very different type of divergence. And again, it's root is in the non-linear version of the recursion undulations that we have. The closest thing to this that we have is when we were calculating the correlation length for the non-linear sigma model where we had something that had a 1 over temperature type of behavior. This is even more complicated.

Now once you know the singular behavior of the correlation length, you would say that in the two dimensional system, the singular part of the free energy should scale like c to the minus 2. Essentially, you break your system into pieces that are of the size correlation length. The number of those pieces is l over xi squared, because you are in two dimensions.

So you would get this. So that says that your singularity of the free energy is something like, I don't know, pi squared 8 4b square root of T minus Tc. Again, not a popular singularity. It's an essential singularity.

And essential singularity is a kind of singular function that no matter how many derivatives you take, as T comes to Tc, there is no singularity. So for example, if I take two derivatives to get the heat capacity, what I would plot as a function of T at Tc should have no signatures. So basically what you would see because if this is that the curve just continues. There is no signature of a transition at this heat capacity.

And indeed, people later on, they did numerical simulations, et cetera. What they find is that the heat capacity actually has kind of smooth peak a little bit later than Tc, which is the location where there's lots and lots of vortex unbinding going, gone. But at Tc itself, there is no signature of a singularity. As far as I know, there's no experimental case with this correlation length as we observed.

So the lesson that we can take from this particular system is that two dimensional system are kind of potentially interesting and different. We had this Mermin-Wagner Theorem that we mentioned in the beginning that said that there should be no true long range order than two dimensions. That is still true.

But despite that, there could be phase transitions with quite observable consequences. Add a particular type of transition in two dimension that we will pursue next lecture-- so I'll give you a preview-- is that of melting. So the prototype of a phase transition you may think of is either liquid gas or a liquid solid.

And you can say, well, you have studied phase transitions to such a degree. Why

not go back and talk about the melting transition, for example? The reason is that the straight melting transition is typically first order. And we've seen that universality and all of those things emerge when you have a diverging correlation length.

So you want to have a place where there is a possible potential for continuous phase transition. And it turns out that melting in two dimensions provides that. So in two dimensions, you could have a bunch of points that could, for example, in a minimum energy configuration at T close to 0 form a triangular lattice.

Now when you go to finite temperature-- as we discussed, again, at the very first lecture-- you will start to have distortions around this. We can describe these distortions to effect of u of x and y. And then go to that appropriate continuum limit that describes the elasticity of these things.

And it is going to look very much like that gradient of theta squared term that we wrote at the beginning, except that since this u is a vector, as we saw, even for an isotropic material, you will have the potential for having multiple elastic constants. But modeled on that, the conclusion that you would have is that as long as it is OK for me to make an expansion that is like the elastic theory-- some kind of a gradient of u expansion-- the conclusion would be that the correlations in u will grow logarithmically as a function of size.

And you will not have too long range of order, but you will have some kind of a power of behavior, such as the one that we have indicated over there. On the other hand, when you go to very high temperature, presumably this whole thing melts. There is no reason to have correlations beyond a few atoms that are close to you. Add so typically at high temperature, correlations will decay exponentially.

So this is low temperature expansion is elastic theory expansion that we have written down has to break down also in this case. And a particular mechanism for its breakdown in two dimensions is to create these topological defects, which in the case of solid will correspond to these location lines that, for example, correspond to adding an addition row of particles here terminating at some point. And we can go through exactly the same kind of story as we had before and conclude that these

dislocations, because of the competition between their energy cost soaring logarithmically and their entropy gain growing logarithmically, we need to unbind at the critical temperature.

And so that provides a mechanism for describing the melting of two-dimensional materials in a language that is very similar to this, except for the complications that have to do with this being a vector rather than the scale of quantity. And so what we find is that these topological charges are different from minus 2 plus 2 pi.

The interactions between them is a particular version of the Coulomb interaction, but that many of the other results go through. And we will get an idea of what happens when the solid melts because of the unbinding of these locations. But there is a puzzle that we will find if we're not melting to a liquid, but into something which is more like a liquid crystal. So [INAUDIBLE] did they discover also something about liquid crystals.