V.C The Niemeijer–van Leeuwen Cumulant Approximation

Unfortunately, the decimation procedure cannot be performed exactly in higher dimensions. For example, the square lattice can be divided into two sublattices. For an RG with $b = \sqrt{2}$, we can start by decimating the spins on one sublattice. The interactions between the four spins surrounding each decimated spin are obtained by generalizing eq.(V.13). If initially $h = g = 0$, we obtain

$$R(\sigma_1', \sigma_2', \sigma_3', \sigma_4') = \sum_{s=\pm 1} e^{Ks(\sigma_1'+\sigma_2'+\sigma_3'+\sigma_4')} = 2\cosh\left[K(\sigma_1' + \sigma_2' + \sigma_3' + \sigma_4')\right]. \qquad \text{(V.28)}$$

Clearly the four spins appear symmetrically in the above expression, and hence are subject to the same two body interaction. This implies that new interactions along the diagonals of the renormalized lattice are also generated, and the nearest neighbor form of the original Hamiltonian is not preserved. There is also a four point interaction, and

$$R = \exp\left[g' + K'(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_4' + \sigma_4'\sigma_1' + \sigma_1'\sigma_3' + \sigma_2'\sigma_4') + K_4'\sigma_1'\sigma_2'\sigma_3'\sigma_4'\right]. \qquad \text{(V.29)}$$

The number (and range) of new interactions increases with each RG step, and some truncating approximation is necessary. Two such schemes are described in the following sections.

One of the earliest approaches was developed by Niemeijer and van Leeuwen (NvL) for treating the Ising model on a *triangular lattice*, subject to the usual nearest neighbor Hamiltonian $-\beta\mathcal{H} = K\sum_{\langle ij\rangle}\sigma_i\sigma_j$. The original lattice sites are grouped into *cells* of three spins (e.g. in alternating up pointing triangles). Labelling the three spins in cell α as $\{\sigma_\alpha^1, \sigma_\alpha^2, \sigma_\alpha^3\}$, we can use a *majority rule* to define the renormalized cell spin as

$$\sigma_\alpha' = \text{sign}\left[\sigma_\alpha^1 + \sigma_\alpha^2 + \sigma_\alpha^3\right]. \qquad \text{(V.30)}$$

(There is no ambiguity in the rule for any odd number of sites, and the renormalized spin is two–valued.) The renormalized interactions corresponding to the above map are obtained from the constrained sum

$$e^{-\beta\mathcal{H}'[\sigma_\alpha']} = \sum_{\{\sigma_\alpha^i \mapsto \sigma_\alpha'\}}{}' e^{-\beta\mathcal{H}[\sigma_\alpha^i]} \quad . \qquad \text{(V.31)}$$

To truncate the number of interactions in the renormalized Hamiltonian, NvL introduced a perturbative scheme by setting $\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$. The unperturbed Hamiltonian

$$-\beta\mathcal{H}_0 = K\sum_{\alpha}\left(\sigma_\alpha^1\sigma_\alpha^2 + \sigma_\alpha^2\sigma_\alpha^3 + \sigma_\alpha^3\sigma_\alpha^1\right), \tag{V.32}$$

involves only *intra–cell interactions*. Since the cells are decoupled, this part of the Hamiltonian can be treated exactly. The remaining *inter–cell interactions* are treated as a perturbation

$$-\mathcal{U} = K\sum_{<\alpha,\beta>}\left(\sigma_\beta^{(1)}\sigma_\alpha^{(2)} + \sigma_\beta^{(1)}\sigma_\alpha^{(3)}\right). \tag{V.33}$$

The sum is over all neighboring cells, each connected by two bonds. (The actual spins involved depend on the relative orientations of the cells.) Eq.(V.31) is now evaluated perturbatively as

$$e^{-\beta\mathcal{H}'[\sigma'_\alpha]} = {\sum_{\{\sigma_\alpha^i \mapsto \sigma'_\alpha\}}}' e^{-\beta\mathcal{H}_0[\sigma_\alpha^i]}\left[1 - \mathcal{U} + \frac{\mathcal{U}^2}{2} - \cdots\right]. \tag{V.34}$$

The renormalized Hamiltonian is given by the cumulant series

$$\beta\mathcal{H}'[\sigma'_\alpha] = -\ln Z_0[\sigma'_\alpha] + \langle\mathcal{U}\rangle_0 - \frac{1}{2}\left(\langle\mathcal{U}^2\rangle_0 - \langle\mathcal{U}\rangle_0^2\right) + \mathcal{O}(\mathcal{U}^3), \tag{V.35}$$

where $\langle\rangle_0$ refers to the expectation values with respect to $\beta\mathcal{H}_0$, with the restriction of fixed $[\sigma'_\alpha]$, and Z_0 is the corresponding partition function.

To proceed, we construct a table of all possible configurations of spins within a cell, their renormalized value, and contribution to the cell energy:

σ'_α	σ_α^1	σ_α^2	σ_α^3	$\exp\left[-\beta\mathcal{H}_0\right]$
$+$	$+$	$+$	$+$	e^{3K}
$+$	$-$	$+$	$+$	e^{-K}
$+$	$+$	$-$	$+$	e^{-K}
$+$	$+$	$+$	$-$	e^{-K}
$-$	$-$	$-$	$-$	e^{3K}
$-$	$+$	$-$	$-$	e^{-K}
$-$	$-$	$+$	$-$	e^{-K}
$-$	$-$	$-$	$+$	e^{-K}

The restricted partition function is the product of contributions from the independent cells,

$$Z_0[\sigma'_\alpha] = \prod_\alpha \left[{\sum_{\{\sigma^i_\alpha \mapsto \sigma'_\alpha\}}}' e^{K(\sigma^1_\alpha\sigma^2_\alpha + \sigma^2_\alpha\sigma^3_\alpha + \sigma^3_\alpha\sigma^1_\alpha)} \right] = \left(e^{3K} + 3e^{-K}\right)^{N/3}. \tag{V.36}$$

It is *independent* of $[\sigma'_\alpha]$, thus contributing an additive constant to the Hamiltonian. The first cumulant of the interaction is

$$-\langle \mathcal{U} \rangle_0 = K \sum_{<\alpha,\beta>} \left[\left\langle \sigma^1_\beta \right\rangle_0 \left\langle \sigma^2_\alpha \right\rangle_0 + \left\langle \sigma^1_\beta \right\rangle_0 \left\langle \sigma^3_\alpha \right\rangle_0 \right] = 2K \sum_{<\alpha,\beta>} \left\langle \sigma^i_\alpha \right\rangle_0 \left\langle \sigma^j_\beta \right\rangle_0, \tag{V.37}$$

where we have taken advantage of the equivalence of the three spins in each cell. Using the table, we can evaluate the restricted average of site spins as

$$\left\langle \sigma^i_\alpha \right\rangle_0 = \left\{ \begin{array}{ll} \dfrac{+e^{3K} - e^{-K} + 2e^{-K}}{e^{3K} + 3e^{-K}} & \text{for} \quad \sigma'_\alpha = +1 \\ \dfrac{-e^{3K} + e^{-K} - 2e^{-K}}{e^{3K} + 3e^{-K}} & \text{for} \quad \sigma'_\alpha = -1 \end{array} \right\} \equiv \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \, \sigma'_\alpha \, . \tag{V.38}$$

Substituting in eq.(V.37) leads to

$$-\beta \mathcal{H}'[\sigma'_\alpha] = \frac{N}{3} \ln\left(e^{3K} + 3e^{-K}\right) + 2K \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \right)^2 \sum_{\langle \alpha\beta \rangle} \sigma'_\alpha \sigma'_\beta + \mathcal{O}(\mathcal{U}^2). \tag{V.39}$$

At this order, the renormalized Hamiltonian involves only nearest neighbor interactions, with the recursion relation

$$K' = 2K \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \right)^2 . \tag{V.40}$$

1. Eq.(V.40) has the following *fixed points:*

(a) The high temperature sink at $K^* = 0$. If $K \ll 1$, $K' \approx 2K(2/4)^2 = K/2 < K$, i.e. this fixed point is *stable*, and has zero correlation length.

(b) The low temperature sink at $K^* = \infty$. If $K \gg 1$, then $K' \approx 2K > K$, i.e. unlike the one dimensional case, this fixed point is also stable with zero correlation length.

(c) Since both of the above fixed points are unstable, there must be at least one stable fixed point at finite $K' = K = K^*$. From eq.(V.40), the fixed point position satisfies

$$\frac{1}{\sqrt{2}} = \frac{e^{3K^*} + e^{-K^*}}{e^{3K^*} + 3e^{-K^*}}, \quad \Longrightarrow \quad \sqrt{2}e^{4K^*} + \sqrt{2} = e^{4K^*} + 3. \tag{V.41}$$

The fixed point value

$$K^* = \frac{1}{4}\ln\left(\frac{3-\sqrt{2}}{\sqrt{2}-1}\right) \approx 0.3356, \tag{V.42}$$

can be compared to the exactly known value of 0.2747 for the triangular lattice.

2. Linearizing the recursion relation around the non-trivial fixed point yields,

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2\left(\frac{e^{4K^*}+1}{e^{4K^*}+3}\right)^2 + 32K^* e^{4K^*}\frac{(e^{4K^*}+1)}{(e^{4K^*}+3)^3} \approx 1.624. \tag{V.43}$$

The fixed point is indeed unstable as required by the continuity of flows. This RG scheme removes 1/3 of the degrees of freedom, and corresponds to $b = \sqrt{3}$. The thermal eigenvalue is thus obtained as

$$b^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K^*}, \quad \Longrightarrow \quad y_t \approx \frac{\ln(1.624)}{\ln(\sqrt{3})} \approx 0.883. \tag{V.44}$$

This can be compared to the exactly known value of $y_t = 1$, for the two dimensional Ising model. It is certainly better than the mean-field (Gaussian) estimate of $y_t = 2$. From this eigenvalue we can estimate the exponents

$$\nu = 1/y_t \approx 1.13 \quad (1), \quad \text{and} \quad \alpha = 2 - 2/y_t = -0.26 \quad (0),$$

where the exact values are given in the brackets.

3. To complete the calculation of exponents, we need the *magnetic eigenvalue* y_h, obtained after adding a magnetic field to the Hamiltonian, i.e. from

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} - h\sum_i \sigma_\alpha^i \quad . \tag{V.45}$$

Since the fixed point occurs for $h^* = 0$, the added term can also be treated perturbatively, and to the lowest order

$$\beta\mathcal{H}' = \beta\mathcal{H}_0 + \langle\mathcal{U}\rangle_0 - h\sum_\alpha \left\langle(\sigma_\alpha^1 + \sigma_\alpha^2 + \sigma_\alpha^3)\right\rangle_0, \tag{V.46}$$

where the spins are grouped according to their cells. Using eq.(V.38),

$$\beta\mathcal{H}' = \ln Z_0 + K'\sum_{<\alpha,\beta>}\sigma'_\alpha\sigma'_\beta - 3h\sum_\alpha\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)\sigma'_\alpha, \tag{V.47}$$

thus identifying the renormalized magnetic field as

$$h' = 3h \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \right). \tag{V.48}$$

In the vicinity of the unstable fixed point

$$b^{y_h} = \left. \frac{\partial h'}{\partial h} \right|_{K^*} = 3 \, \frac{e^{4K^*} + 1}{e^{4K^*} + 3} = \frac{3}{\sqrt{2}}, \tag{V.49}$$

and

$$y_h = \frac{\ln\left(3/\sqrt{2}\right)}{\ln\left(\sqrt{3}\right)} \approx 1.37. \tag{V.50}$$

This is lower than the exact value of $y_h = 1.875$. (The Gaussian value of $y_h = 2$ is closer to the correct result in this case.)

4. NvL carried out the approach to the second order in $\mathcal{U}$. At this order two additional interactions over further neighbor spins are generated. The recursion relations in this three parameter space have a non-trivial fixed point with one unstable direction. The resulting eigenvalue of $y_t = 1.053$, is tantalizingly close to the exact value of 1, but this agreement is probably accidental.

V.D The Migdal–Kadanoff Bond Moving Approximation

Consider a $b = 2$ RG for the Ising model on a square lattice, in which every other spin along each lattice direction is decimated. As noted earlier, such decimation generates new interactions between the remaining spins. One way of overcoming this difficulty is to simply remove the bonds not connected to the retained spins. The renormalized spins are then connected to their nearest neighbors by two successive bonds. Clearly after decimation, the renormalized bond is given by the recursion relation in eq.(V.18), characteristic of a one dimensional chain. The approximation of simply removing the unwanted bonds weakens the system to the extent that it behaves one dimensionally. This is remedied by using the unwanted bonds to strengthen those that are left behind. The spins that are retained are now connected by a pair of double bonds (of strength $2K$), and the decimation leads to

$$K' = \frac{1}{2} \ln \cosh(2 \times 2K) \, . \tag{V.51}$$

1. Fixed points of this recursion relation are located at

(a) $K^* = 0$: For $K \ll 1$, $K' \approx \ln(1 + 8K^2)/2 \approx 4K^2 \ll K$, i.e. this fixed point is stable.

(b) $K^* \to \infty$: For $K \gg 1$, $K' \approx \ln(e^{4K}/2)/2 \approx 2K \gg K$, indicating that the low temperature sink is also stable.

(c) The domains of attractions of the above sinks are separated by a third fixed point at

$$e^{2K^*} = \frac{e^{4K^*} + e^{-4K^*}}{2}, \quad \Longrightarrow \quad K^* \approx 0.305, \tag{V.52}$$

which can be compared with the exact value of $K_c \approx 0.441$.

2. Linearizing eq.(V.51) near the fixed point gives

$$b^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2\tanh 4K^* \approx 1.6786, \quad \Longrightarrow \quad y_t \approx 0.747, \tag{V.53}$$

compared to the exact value of $y_t = 1$.

The bond moving procedure can be extended to *higher dimensions*. For a hypercubic lattice in d-dimensions, the bond moving step strengthens each bond by a factor of 2^{d-1}. After decimation, the recursion relation is

$$K' = \frac{1}{2}\ln\cosh\left[2 \times 2^{d-1}K\right]. \tag{V.54}$$

The high and low temperature sinks at $K^* = 0$ and $K^* \to \infty$, are stable, since

$$K \ll 1, \quad \Longrightarrow \quad K' \approx \frac{1}{2}\ln(1 + 2^{2d-1}K^2) \approx 2^{2(d-1)}K^2 \ll K, \tag{V.55}$$

and

$$K \gg 1, \quad \Longrightarrow \quad K' \approx \frac{1}{2}\ln\frac{e^{2^d K}}{2} \approx 2^{d-1}K \gg K. \tag{V.56}$$

(Note that the above result correctly identifies the lower critical dimension of the Ising model, in that the low temperature sink is stable only for $d > 1$.) The intervening fixed point has an eigenvalue

$$2^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{d-1}\tanh\left(2^d K^*\right). \tag{V.57}$$

The resulting values of $K^* \approx 0.065$ and $y_t \approx 0.934$ for $d = 3$, can be compared with the known values of $K_c \approx 0.222$ and $y_t \approx 1.59$ on a cubic lattice. Clearly the approximation gets worse at higher dimensions. (It fails to identify an upper critical dimension, and as $d \to \infty$, $K^* \to 2^{2(1-d)}$ and $y_t \to 1$.)

The Migdal–Kadanoff scheme can also be applied to more general spin systems. For a one dimensional model described by the set of interactions $\{K\}$, the transfer matrix method in eq.(V.27) gives the recursion relations as

$$T'_b(\{K'\}) = T(\{K\})^b.$$

For a d-dimensional lattice, the bond moving step strengthens each bond by a factor of b^{d-1}, and the generalized Migdal–Kadanoff recursion relations are

$$T'_b(\{K'\}) = T(\{b^{d-1}K\})^b. \tag{V.58}$$

The above equations can be used as a quick way of estimating phase diagrams and exponents. The procedure is exact in $d = 1$, and does progressively worse in higher dimensions. It thus compliments mean–field (saddle point) approaches that are more reliable in higher dimensions. Unfortunately, it is not possible to develop a systematic scheme to improve upon its results. The RG procedure also allows evaluation of free energies, heat capacities, and other thermodynamic functions. One possible worry is that the approximations used to construct RG schemes may result in unphysical behavior, e.g. negative values of response functions C and χ. In fact most of these recursion relations (e.g. eq.(V.58)) are exact on *hierarchical* (Berker) lattices. The realizability of such lattices ensures that there are no unphysical consequences of the recursion relations.

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