8.324 Relativistic Quantum Field Theory II

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Lecture 20

5: THE RENORMALIZATION GROUP

The renormalization program has been shown to be a success operationally:

1.   For a renormalizable theory, all ultraviolet divergences can be absorbed into redefinitions of bare quantities, which are not observable. What is meant by the set of physical observables is unambiguous: it is well-defined in terms of a small number of physical couplings, masses and so forth.

2.   The requirement of renormalizability is a strong constraint on the physical theories we should consider. Theories satisfying this constraint included quantum electrodynamics, Yang-Mills theories and the standard model.

Conceptually, we want to understand why it works. Many of the older generation of physicists did not accept renormalization when it was first introduced operationally. There are several difficulties with this:

1.   The process is perturbative in nature: it has to be carried out order-by-order in perturbation theory. Also, when carrying out perturbation theory, with a Lagrangian of the form

$$\mathscr{L} = -\frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 - \mathscr{L}_{ct}, \tag{1}$$

     for example, with $\mathscr{L}_{ct} = -\frac{1}{2}A\left(\partial\phi\right)^2 - \frac{1}{2}Bm^2\phi^2 - \frac{1}{3!}C\phi^3$. The coefficients A, B and C are found to be divergent, yet they are considered to be small quantities in the perturbation expansion.

2.   Non-renormalizable terms are generated by loop corrections, such as the anomalous magnetic moment term, $\bar{\psi}\gamma^{\mu\nu}\psi F_{\mu\nu}$. Physically, we want to understand why this is not a bare term.

3.   Conceptually, it is troublesome to assume that a particular theory like quantum electrodynamics should be valid to arbitrarily high energies.

5.1: WILSON'S APPROACH TO FIELD THEORIES

Let us consider Wilson's basic starting points, which should be viewed as the postulates of the theory.

1.   A quantum field theory should be considered as an effective field theory, valid only in a certain range of energies. In particular, it should be defined with an ultraviolet cut-off Λ_0. For example, in quantum electrodynamics, we should choose $\Lambda_0 \sim 1\text{TeV}$, and we should only expect the theory to be valid for $E \ll \Lambda_0$.

2.   Physical phenomena at a certain energy scale E are most appropriately described by the degrees of freedom at that scale. For example, for a scalar field

$$\phi(x) \leftrightarrow \phi(k), \quad k \in (0, \infty), \tag{2}$$

     the most relevant degrees of freedom should be $\phi(k \sim E)$.

At the moment, these notions have only been expressed philosophically. The key is to actually implement these ideas procedurally. We will use the example of a single real scalar field for illustration, working in Euclidean signature. The simplest cut-off procedure is to restrict

$$|k| < \Lambda_0, \tag{3}$$

though other cut-off procedures can be considered. The basics ingredients of Wilson's approach are as followed:

1.   Start with a bare Lagrangian defined at a scale Λ_0 :

$$
\begin{aligned}
\mathscr{L}\left[\Lambda_0\right] & = \frac{1}{2}\left(\partial\phi\right)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{g_4^{(0)}}{4!}\phi^4 + \frac{g_6^{(0)}}{6!}\phi^6 + \ldots \\
& \equiv \mathscr{L}_0 + \mathscr{L}_{int},
\end{aligned}
$$

with $\mathscr{L}_0 \equiv \frac{1}{2}\left(\partial\phi\right)^2$, $\mathscr{L}_{int} \equiv \sum_i g_i^{(0)}O_i$, $O_n \equiv \frac{1}{n!}\phi^n$. Here, $\mathscr{L}\left[\Lambda_0\right]$ is only valid below a cut-off scale Λ_0. The bare couplings $\left\{g_i^{(0)}\right\}$ encode the unknown physics above Λ_0. $\mathscr{L}\left[\Lambda_0\right]$ includes all the interactions consistent with Lorentz symmetry and an additional discrete symmetry under $\phi \leftrightarrow -\phi$. That is, $\{O_i\}$ is a complete set of local operators consistent with a $\mathbb{Z}_2$ symmetry. If

$$
[O_i] = \Delta_i, \tag{4}
$$

then

$$
\left[g_i^{(0)}\right] = d - \Delta_i \equiv \delta_i, \tag{5}
$$

and so we introduce dimensionless couplings by

$$
\lambda_i^{(0)} \equiv g_i^{(0)}\Lambda_0^{-\delta_i}. \tag{6}
$$

We assume that $\lambda_i^{(0)} \sim O(1)$ for all i, treating all couplings on an equal footing, including the non-renormalizable ones; those satisfying $\delta_i < 0$.

2.   The physics below the cut-off scale Λ_0 is fully captured by the path integral,

$$
Z\left[J\right] = \int_{k<\Lambda_0} \mathfrak{D}\phi(k)\, e^{-S[\Lambda_0]-\int d^dx\, J\phi}, \tag{7}
$$

where $S\left[\Lambda_0\right] = \int d^dx\, \mathscr{L}\left[\Lambda_0\right]$. The integration is only over $\phi(k)$ with $k < \Lambda_0$.

3.   If we are interested in physics at the same energy scale $E < \Lambda_0$, we can integrate out degrees of freedom, for example $\phi(k)$ for $\Lambda < k < \Lambda_0$, where Λ is a new cut-off scale, $\Lambda \sim E$. More explicitly, we write

$$
\phi(k) = \phi_\Lambda(k) + \tilde{\phi}(k), \tag{8}
$$

where $\phi_\Lambda(k)$ has support for $k \in (0, \Lambda)$, and $\tilde{\phi}(k)$ has support for (Λ, Λ_0). Then,

$$
\int_{k<\Lambda_0} \mathfrak{D}\phi(k) = \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k) \int_{\Lambda<k<\Lambda_0} \mathfrak{D}\tilde{\phi}(k), \tag{9}
$$

and hence we have

$$
\begin{aligned}
Z\left[J\right] & = \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k)\, e^{\int d^dx\, J\phi_\Lambda} \int_{\Lambda<k<\Lambda_0} \mathfrak{D}\tilde{\phi}(k)\, e^{-S[\phi_\Lambda+\tilde{\phi},\Lambda_0]-\int d^dx\, J\tilde{\phi}} \\
& = \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k)\, e^{-S[\phi_\Lambda,\Lambda]-\int d^dx\, J\phi_\Lambda},
\end{aligned}
$$

where $S\left[\phi_\Lambda, \Lambda\right]$ is the effective action at the new cut-off scale Λ, for the degrees of freedom contained in ϕ_Λ. $S\left[\phi_\Lambda, \Lambda\right]$ encodes the effects of all the high energy degrees of freedom, $\phi(k)$ with $k < \Lambda_0$ on the degrees of freedom below the new scale Λ. It captures all of the essential physics at the scale $k \sim \Lambda$. $S\left[\phi_\Lambda, \Lambda\right]$ is, of course, very complicated. We can parameterize it as

$$
S\left[\phi_\Lambda, \Lambda\right] = \sum_{n=0}^{\infty} \int^{\Lambda} \left(\prod_{i=1}^{n} d^dk_i\right) \delta^{(d)}(k_1 + k_2 + \ldots + k_n) \times F_n(k_1, \ldots, k_n)\phi_\Lambda(k_1)\ldots\phi_\Lambda(k_n). \tag{10}
$$

Here, the δ-function is a consequence of translational symmetry. Expanding $F_n(k_1, \ldots, k_n)$ as a power series of k_i, we can rewrite

$$
\mathscr{L}\left[\phi_\Lambda, \Lambda\right] = \frac{1}{2}Z(\partial\phi)^2 + \sum_i g_i O_i, \tag{11}
$$

where O_i is a complete set of local operators of ϕ_n allowed by symmetries. Here,

$$Z(\{g_i^{(0)}\}, \Lambda_0; \Lambda) \tag{12}$$

is the field renormalization from integrating out the degrees of freedom $\tilde{\phi}$, and

$$\{g_i(\{g_i^{(0)}\}, \Lambda_0; \Lambda)\} \tag{13}$$

are the physical couplings at the scale Λ. For example,

$$F_2(k_1, k_2) = F_2(k; \Lambda) = c_o + c_1 k^2 + \dots \tag{14}$$

where we can write F_2 a function of a single $k < \Lambda$ because of the $\delta-$function in (10). In coordinate space, this contribute to the effective action can be written as

$$c_0 \phi^2 + c_1 (\partial \phi)^2 + \dots. \tag{15}$$

So, after changing scale, the Lagrangian

$$\mathscr{L}_0[\Lambda_0] = \frac{1}{2}(\partial \phi)^2 + \sum_i g_i^{(0)} O_i \tag{16}$$

is replaced by

$$\mathscr{L}[\Lambda] = \frac{1}{2}Z(\partial \phi)^2 + \sum_i g_i O_i. \tag{17}$$

At first sight, Wilson's approach seems very complicated: $S[\phi_\Lambda, \Lambda]$, which is parameterized by an infinite number of interactions

$$\{g_i(\{g_i^{(0)}\}, \Lambda_0; \Lambda)\}, \tag{18}$$

depends on an infinite number of bare couplings,

$$\{g_i^{(0)}\} \tag{19}$$

at the scale Λ_0. To this extent, there is no predictive power. But it turns out there is some additional structure in this approach which makes it precisely the right way of thinking about field theories. First, a simple dimensional analysis can be used to give us some indication of this. We return to the bare action,

$$\begin{aligned} S[\Lambda_0] &= \int d^d x \, \frac{1}{2}(\partial \phi)^2 + \sum_i g_i^{(0)} O_i \\ &= \sum_i S_i, \end{aligned}$$

where $g_i^{(0)} \equiv \lambda_i^{(0)} \Lambda_0^{-\delta_i}$, and we suppose that we are interested in a process at an energy scale E. Then, if we assume all dimensional quantities are controlled by E; that is,

$$k \sim E, \quad \phi \sim E^{\frac{d-2}{2}}, \quad O_i \sim E^{\Delta_i}, \tag{20}$$

then we have

$$\begin{aligned} S_i &\sim \int d^d x \, g_i^{(0)} O_i &\sim \quad E^{\Delta_i - d} g_i^{(0)} = E^{-\delta_i} g_i^{(0)} \\ &&\sim \quad \lambda_i^{(0)} \left(\frac{E}{\Lambda_0}\right)^{-\delta_i}, \end{aligned}$$

where we have assumed that $\lambda_i^{(0)} \sim O(1)$, and we have that $\delta_i \equiv d - \Delta_i$. Then, for $E \ll \Lambda_0$, there are three cases:

1. $\delta_i > 0$, or $\Delta_i > d$: irrelevant coupling, $S_i \ll 1$ and the effect becomes negligibile for $E \ll \Lambda_0$.

2. $\delta_i = 0$, or $\Delta_i = d$: marginal coupling, $S_i \sim O(1)$.

3. $\delta_i < 0$, or $\Delta_i < d$: relevant coupling, $S_i \gg 1$.

So, for $E \ll \Lambda_0$, the non-renormalizable (irrelevant) bare couplings become suppressed. In particular, in the limit $\frac{E}{\Lambda_0} \longrightarrow 0$, or equivalently, $\Lambda_0 \longrightarrow \infty$, their contributions vanish. Therefore, the low energy physics only depends on a small number of renormalizable couplings, the marginal and relevant couplings, satisfying $\delta_i \geq 0$, even though in the bare Lagrangian, we took all couplings to appear at comparable strength. More explicitly, if we write

$$\left\{ g_i^{(0)} \right\} = \left\{ \rho_a^{(0)} \right\} + \left\{ \kappa_b^{(0)} \right\} \tag{21}$$

where $\left\{ \rho_a^{(0)} \right\}$ are the renormalizable and marginal couplings, and $\left\{ \kappa_b^{(0)} \right\}$ are the non-renormalizable couplings, then in the limit $\frac{\Lambda}{\Lambda_0} \longrightarrow 0$,

$$
\begin{aligned}
g_i &= g(\left\{ g_i^{(0)} \right\}, \Lambda_0; \Lambda) \\
&\longrightarrow g(\left\{ \rho_a^{(0)} \right\}, \Lambda_0; \Lambda).
\end{aligned}
$$

This crude dimensional analysis will be made precise in later lectures.

Remarks:

1.   In the Wilsonian approach, renormalization comes from integrating out the short-distance degrees of freedom.

2.   No ultraviolet divergences are ever encountered.

3.   There are underlying physical reasons for distinguishing renormalizable and non-renormalizable couplings.

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