Problem Set #1   8.311

Problem 1

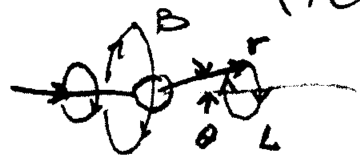
a) $\nabla \times B = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}(j + j_d)$

$0 = \nabla\cdot(\nabla \times B) = \frac{4\pi}{c} \nabla\cdot\underbrace{(j + j_d)}_{J} \rightarrow \nabla J = 0$

div·curl = 0

b) $\nabla \cdot E = 4\pi\rho \xrightarrow{\text{spherical symmetry}} \vec{E} = \frac{q}{r^2}\hat{r}, \quad j_d = \frac{\dot{q}}{4\pi r^2}\hat{r}$

c) From axial symmetry, expect circular field B lines



$\oint B \cdot dl = \frac{4\pi}{c} J_{total}$

To obtain B at a point distance r away from the origin, angle θ w.r.p.t. the wire axis, consider the circular loop L shown in drawing

$2\pi\rho_L B = \frac{4\pi}{c}\frac{1}{4\pi}\Omega\dot{q}$ $\Omega = 2\pi(1-\cos\theta)$

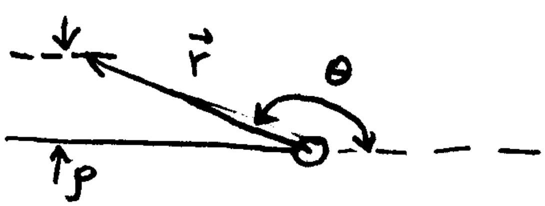
loop radius   loop solid angle $\rho_L = r\sin\theta$

$$B(r,\theta) = \frac{2\pi(1-\cos\theta) I}{2\pi c r \sin\theta} = \underline{\underline{\frac{I}{cr}\tan\theta/2}}$$

Near the wire, $\theta \rightarrow \pi$, $\cos\theta \rightarrow -1$

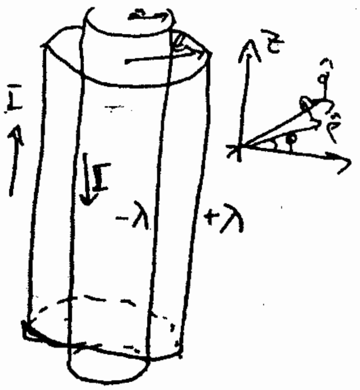
$$B(r,\theta) \approx \frac{2I}{cr\sin(\pi-\theta)} = \frac{2I}{c\rho}$$

(matches the infinite wire result)



Problem 2

Applying Gauss's and Ampère's laws one can find $\vec{E}$ and $\vec{B}$.



$$|\vec{B}|\cdot 2\pi\rho = \frac{4\pi}{c} I \Rightarrow |\vec{B}| = \frac{2I}{\rho c} \quad (a<\rho<b)$$

$$\vec{B} = -\frac{2I}{\rho c}\hat{\varphi} \quad (a<\rho<b)$$

$$\vec{B} = 0 \quad \text{if} \quad \rho<a \text{ or } \rho>b$$

$$|\vec{E}|\cdot l\cdot 2\pi\rho = 4\pi l\cdot\lambda \Rightarrow |\vec{E}| = \frac{2\lambda}{\rho}(a<\rho<b)$$

$$\vec{E} = -\frac{2\lambda}{\rho}\cdot\hat{\rho} \quad (a<\rho<b) \qquad \vec{E}=0 \quad (\rho<a \text{ or } \rho>b)$$

$$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{H}) = \frac{c}{4\pi}\cdot\frac{2\lambda}{\rho}\cdot\frac{2I}{\rho c}[\hat{\rho}\times\hat{\varphi}] = \frac{\lambda I}{\pi}\frac{1}{\rho^2}\cdot\hat{z}$$

Total energy flux:

$$F = \int\vec{S}\cdot d\vec{A} = 2\pi\int_a^b \frac{\lambda I}{\pi}\frac{1}{\rho^2}\cdot\rho\, d\rho = \underline{\lambda I \ln b^2/a^2 = 2\lambda I \ln\frac{b}{a}}$$

Resistor is connected:

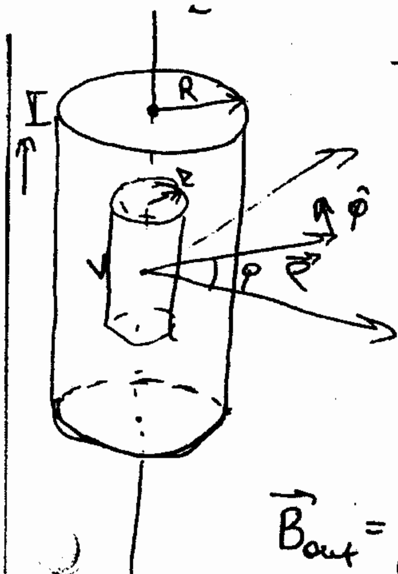
given λ, potential difference between the cylinders is:

$$U = \int_a^b \frac{2\lambda}{\rho}d\rho = 2\lambda\ln b/a$$

In terms of R: $F = 2\lambda I \ln b/a = U\cdot I = I^2 R$

and is equal to the energy dissipated in the resistor $\cdot R$.

Problem 3



$$\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = \frac{I}{\pi R^2 \sigma} \hat{z}$$

$$\vec{B}_{in} = \frac{2I}{cR^2} \rho \hat{\varphi}$$

$$\vec{S}_{in} = \frac{c}{4\pi}\left[\vec{E}_{in} \times \vec{B}_{in}\right] = \frac{c}{4\pi} \cdot \frac{I}{\pi R^2 \sigma} \cdot \frac{2I}{cR^2} \rho \cdot [\hat{z} \times \hat{\varphi}] =$$

$$= -\frac{I^2 \rho}{2\pi^2 R^4 \sigma} \hat{\rho}$$

$$\vec{B}_{out} = \frac{2I}{c\rho}\hat{\varphi} \Rightarrow \vec{S}_{out} = -\frac{I^2}{2\pi^2 R^4 \sigma \rho} \cdot \hat{\rho}$$

Consider cylinder of radius $r < R$: (of unit length)

The rate of dissipation inside the cylinder is

$$\int_V \vec{J} \cdot \vec{E}\, d^3x = \sigma |\vec{E}|^2 \cdot \pi r^2 = \frac{I^2 r^2}{\pi R^4 \sigma}$$

Energy flux inside the cylinder:

$$\int \vec{S}_{in}\, d\vec{A} = -2\pi r |\vec{S}_{in}| = -2\pi r \cdot \frac{I^2 r}{2\pi^2 R^4 \sigma} = -\frac{I^2 r^2}{\pi R^4 \sigma}$$

We see that $\int_V \vec{J} \cdot \vec{E}\, d^3x + \int_S \vec{S}\, d\vec{A} = 0$,

i.e. energy is conserved

(Note: $\vec{E}, \vec{B}$ are constant in time $\Rightarrow \frac{dW_{field}}{dt} = 0$)

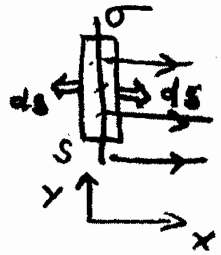
Outside:

$$\int \vec{S}_{out}\, d\vec{A} = -\frac{I^2}{\pi R^2 \sigma}$$

which does not depend on R !

Problem 4

a)



$E = 4\pi\sigma$

$$T_{ij} = \frac{E^2}{8\pi}\delta_{ij} - \frac{E_i E_j}{4\pi} = \begin{pmatrix} -E^2/8\pi & 0 & 0 \\ 0 & E^2/8\pi & 0 \\ 0 & 0 & E^2/8\pi \end{pmatrix}$$

$$F_i = -\oint_S T_{ij}\, ds_j = -T_{i1} A$$

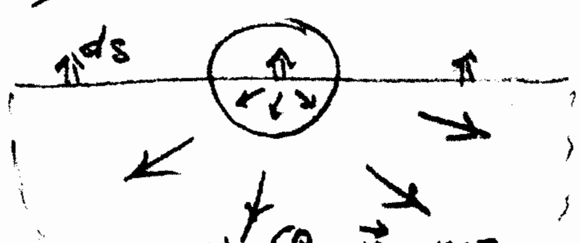
$\underline{\text{Force/Area} = \frac{E^2}{8\pi}\hat{x}}$

To obtain force from $F = qE$, we need E_{ext} which is due to all the charges away from the surface element: $E_{ext} = \frac{1}{2}\left(E_{x>0} + E_{x<0}\right) = \frac{E}{2}$

Then $dF = E_{ext}\,\sigma\, dA = \frac{E}{2}\frac{E}{4\pi}\, dA = \underline{\frac{E^2}{8\pi}\, dA}$

(agrees with the above)

b) calculate force on the lower hemisphere:



By symmetry, $\vec{F}_{total} \parallel \hat{z}$

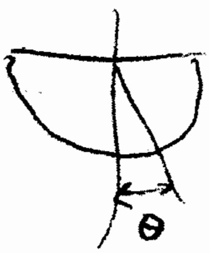
$$\vec{E} = \begin{cases} \frac{Q}{R^3}\vec{r}, & r<R \\ \frac{Q}{R^2}\hat{r}, & r>R \end{cases} \qquad F_z = \int \left(-\frac{E^2}{8\pi}\right) dA \Big|_{z=0}$$

$$-F_z = \int_0^R \frac{Q^2}{8\pi R^6} r^2 2\pi r\,dr + \int_R^\infty \frac{Q^2}{8\pi r^4} 2\pi r\,dr = \left(\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}\right)\frac{Q^2}{R^2}$$

$$\underline{\vec{F} = -\frac{3}{16}\frac{Q^2}{R^2}\hat{z}}$$

Direct calculation: $\vec{F} = \int_{z<0} \rho \vec{E}\, dv$

$$-F_z = \int_0^R \int_0^{\pi/2} \frac{\rho Q}{R^3} r \cos\theta \; 2\pi \sin\theta\, d\theta\, r^2 dr = \frac{\rho Q R}{4} \cdot 2\pi \cdot \frac{1}{2};$$

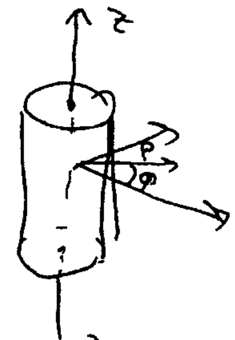


$$-F_z = \frac{3}{4\pi}\frac{Q^2}{R^2}\frac{\pi}{4} = \frac{3}{16}\frac{Q^2}{R^2}$$

$$\underline{\vec{F} = -\frac{3}{16}\frac{Q^2}{R^2}\hat{z}}$$

(agrees with the above)

Problem 5

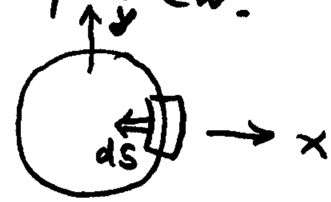


a) $\vec{B} = \frac{4\pi I N}{c}\hat{z}$

$$T_{ij} = -\frac{1}{4\pi}B_i B_j + \frac{1}{8\pi}\delta_{ij}B^2 = \begin{pmatrix} \frac{1}{8\pi}B^2 & 0 & 0 \\ 0 & \frac{1}{8\pi}B^2 & 0 \\ 0 & 0 & -\frac{1}{8\pi}B^2 \end{pmatrix}$$

Due to the symmetry the force does not depend on the φ and z.
Let us choose $\vec{dS} \parallel -\vec{e}_x$

Top view:



$$F_i = -\sum_j T_{ij}\, dS_j = -T_{xx}\vec{e}_x = -\frac{B^2}{8\pi}\cdot\hat{x} \Rightarrow$$

$$\Rightarrow \vec{F} = +2\pi\left(\frac{IN}{c}\right)^2\hat{x}$$

$F = 2\pi\left(\frac{IN}{c}\right)^2$, directed along $\hat{\rho}$ and points outwards

b)

b) For the Lorentz force on an element of the wire $d\vec{l}$:

$$dF_L = \frac{1}{c} I \, dl \, B_{ext} = \frac{1}{2c} I \, dl \, B$$

the force per unit area is

$$F = n \frac{dF_L}{dl} = \frac{n I B}{2c} = \underline{\frac{2\pi I^2 n^2}{c^2}}$$

Note: the field $\vec{B}$ in the vicinity of the surface with current can be represented as a sum of the field $\vec{B}_{surf}$ produced by a small element of the surface and the field $\vec{B}_{ext}$ due to the rest of the solenoid.

Only $\vec{B}_{ext} = \frac{1}{2} \vec{B}$ contributes to $\vec{F}_L$