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**PROFESSOR:** OK, in that case, let's get going. I want to begin by just summarizing where we left off last time, because we're still to some extent in the middle of the same discussion.

We were talking about the gravitational effects of a completely homogeneous universe that fills all of space. And you recall that Newton had concluded that such a system would be stable. But I was arguing that such a system would not be stable, even given the laws of Newtonian mechanics.

And we discussed a few of those arguments last time. We discussed, for example, Gauss's law formulation of Newton's law of gravity. And I remind you that it's a very simple derivation to go from Newton's law of gravity, as Newton stated it as an acceleration at a distance, to Gauss's law.

What you do is you just show that if the acceleration of gravity is given by Newton's law, then for any one particle creating such a gravitational field, Gauss's law holds. This integral is either equal to 0 or minus  $4\pi GM$ , depending on whether the surface that you're integrating over encloses or does not enclose the charge or the mass.

And once you know it for one mass, Newton tells us that for many masses you just add the forces as vectors. And that means that you'll be adding up these integrals for each of the particles. And you're lead automatically to the expression we have here. So really, it follows very directly from Newton's law of gravity.

On the other hand, if we apply this formulation to an infinite distribution of mass, if Newton was right and there were no forces, that would mean that little  $g$ , the acceleration of gravity, would be 0 everywhere. And then, the integral on the left

would be 0. But the term on the right is clearly not 0, if we have a volume with some nontrivial size that includes some mass. So this formulation of Newton's law of gravity clearly shows that an infinite distribution of mass could not be static.

Further, I showed you there's another formulation, more modern, of Newton's law of gravity in terms of what's called Poisson's equation. This was really shown for the benefit of people who know it. If you don't it, don't worry about it. We won't need it.

But it's another way of formulating the law of gravity by introducing a gravitational potential,  $\phi$ , and writing the acceleration of gravity as minus the gradient of  $\phi$ . That just defines  $\phi$ . And then, you can show that  $\phi$  obeys Poisson's equation,  $\nabla^2 \phi$  is equal to  $4\pi G$  times  $\rho$ , where  $\rho$  is the mass density.

And again, one can see immediately that this does not allow a static distribution of mass. If the distribution of mass was static, that would mean that  $\mathbf{g}$  vector was 0. That would mean that gradient  $\phi$  was 0. That would mean that  $\phi$  was a constant. And if  $\phi$  is a constant,  $\nabla^2 \phi$  is 0, and that's inconsistent with the Poisson equation.

I might further add-- I don't think I said this last time-- that from a modern perspective, equations like Poisson's equation are considered more fundamental than Newton's original statement of the law of gravity as an action at a distance. In particular, when one wants to generalize Newton's law, for example, to general relativity, Einstein started with Poisson's equation, not with the force law at a distance.

And there's nothing like the force law at a distance and the theory of general relativity. General relativity is formulated in a language very similar to the language of Poisson's equation. The key idea that underlies this distinction is that all the laws of physics that we know of can be expressed in a local way.

Poisson's equation is a local equation. That's just a differential equation that holds at each point in space and doesn't say anything about how something at one point in space affects something far away. That happens as a consequence of this

equation. But it happens as a consequence of solving the equation. It's not built into the equation to start with.

Continuing. We then discussed what goes on if one does just try to add up the forces using Newton's law and action at a distance. And what I argued is that it's a conditionally convergent integral, which means it's the kind of integral which has the property that it converges. But it can converge to different things depending on what order you add up the different parts of the integral. And we considered two possible orderings for adding up the mass.

In all cases, what we're talking about here is just a single location,  $P$ . And you can't tell the slide is filled with mass, but think of that light cyan as mass. It fills the entire slide and fills the entire universe in our toy problem here.

So we're interested in calculating the force on some point,  $P$ , in the midst of an infinite distribution of mass. And the only thing that we're going to do differently in these two calculations is we're going to add up that mass in a different order.

And if we add up the mass ordered by concentric shells about  $P$ , each consecutive shell clearly contributes 0 to the force at  $P$ . And therefore, the sum in the limit, as we go out to infinity, will still be 0. So for this case, we get  $g$  equals 0. We do get no acceleration at the point,  $P$ , as long as we add up all the masses in that way.

But there's nothing in Newton's laws that tell us what order to add up the forces. Newton just tells you that each mass creates a  $1$  over  $r$  squared force, and it's a vector. And Newton says to add the vectors.

Normally, addition of vectors commutes. It doesn't matter what order you add them in. But what we're going to be finding here is that it does matter what order. And therefore, the answer is ambiguous.

And to see this, we'll consider a different ordering. Instead-- we'll still use spherical shells, because that's the easiest thing to think about. We could try to do something else, but it's much harder to use any other shape.

But this time, we'll consider spherical shells that are centered around a different point. And we'll call the point that the spherical shells are centered around, Q. We're still trying to calculate the force at the point, P, due to the infinite mass distribution that fills space-- so we're doing the same problem we did before-- but we're going to add up the contributions in a different order.

And we discussed last time that when we do that, it turns out that all the mass inside the sphere, centered at Q out to the radius of P, contributes to the acceleration at P. And all mass outside of that, could be divided into concentric shells, where the point, P, is inside. And there's no force inside a concentric shell. So all of the rest of the mass contributes zilch.

And the answer you get is then simply the answer that you would have for the force of a point mass located at Q, whose mass was the total mass in that shaded region. And clearly it's nonzero.

And furthermore, clearly, we can-- by choosing different points, Q-- make this anything we want. We can make it bigger by putting the point, Q, further away; and it always points in the direction of Q. Yeah, points in the direction of Q. So we can let it point in any direction we want by putting the point, Q, anywhere we want.

So depending on how we add up the contributions, we can get any answer we want. And that's a fundamental ambiguity in trying to apply Newton using only the original statement of the law of gravity as Newton gave it. OK.

So the conclusion, here, is that the action at a distance description is simply ambiguous. Descriptions by Gauss's law, or Poisson's law, tell us that the system cannot be static. And we'll soon try to figure out exactly how we do expect it to behave.

Now, I still want come back to one argument, which was really the argument that persuaded Newton in the first place. Newton said that if we want to calculate the acceleration of a certain point in this infinite mass distribution, we have a symmetry problem. All directions from that point looking outward look identical. If there's going

to be an acceleration acting on any point, what could possibly determine the direction that the acceleration will have.

So that's the symmetry argument, which is a very sticky one. It sounds very convincing, in Newton's reasoning. There could be no acceleration, simply because there's no preferred direction for the acceleration to point.

To convince Newton that that's not a valid argument would probably be hard. And I don't know if we could succeed in convincing him or not. Don't get the chance to try.

But if we did have a chance to try, what we would try to explain to him is that the acceleration is usually measured relative to an inertial frame. That's how Newton always described it. And to Newton, there was a unique inertial frame-- unique up to changes in velocity-- determined by the frame of the fixed stars.

That was the language that Newton used. And that defined his inertial frame. And all of his laws of physics were claimed to hold in this inertial frame.

On the other hand, if all of space is filled with matter and it's all going to collapse as we're claiming, there isn't any place to have any fixed stars. So the whole idea of an inertial frame really disappears. There's no object, which one can think of as being at rest or being non-accelerating with respect to any would-be inertial frame.

So in the absence of an inertial frame, one really has to admit that all accelerations, just like all velocities, have to be measured as relative accelerations. We could talk about the acceleration of one mass relative to another. But we can't talk about what the absolute acceleration is of a given mass, because we don't have an inertial frame with which to compare the acceleration of that object.

So when all accelerations are relative, then there are more options here. And it turns out that the right option-- the one that we'll eventually deduce-- is an option that looks similar to Hubble's law.

Hubble's law is a law about velocities. And it says that from the point of view of any observer, all the other objects will look like they're moving radially outward from that

observer. And in spite of the fact that that description makes it sound, emphatically, like the observer you're talking about is special, you can transform to the frame of any other observer and thus seeing exactly the same thing.

So having one observer seeing all the other velocities moving outward from him does not violate homogeneity. It does not violate any of the symmetries that we're trying to incorporate in the system. And the same thing is true for acceleration. So I'm not going to try to show it now. We will be showing it in the course of our upcoming calculations.

But in the kind of collapsing universe that we're going to be describing, any observer can consider himself or herself to be non-accelerating. And then, that observer would see all the other particles accelerating directly towards her. And although that description makes it sound like the person at the center is special, it's not true.

You can transform to any other observer's frame. And each observer can regard himself as being non-accelerating and would see all the other objects accelerating radially towards him.

OK. So now we are ready to go on and try to build a mathematical model, which will tell us how a uniform distribution of mass will behave. Now in doing this, we first would like to tame the issue of infinities. And we are going to do that by starting out with a finite sphere. And then at the very end, we'll let the size of that sphere go to infinity.

So our goal is to build a mathematical model of our toy universe. And what we want to do is to incorporate the three features that we discussed, isotropy, homogeneity, and Hubble's law. And we're going to build this as a mechanical system, using the laws of mechanics as we know them.

We're, in fact, going to be using a Newtonian description. But I will assure you that although we are using a Newtonian description, the answer that we'll get will in fact be the exact answer that we would have gotten with general relativity. And we'll talk later about why that's the case.

But we're not wasting our time doing only an approximate calculation. This actually is a completely valid calculation, which gives us, in the end, entirely the correct answer.

So we're going to model our universe by introducing a coordinate system. I'm now just going to really re-draw the picture that's up there. But if I draw it down here, I can point to it better.

We're going to imagine starting at our toy universe as a finite sized sphere of matter. And we're going to let  $t_{sub\ i}$  be equal to the time of the initial picture.  $t_{sub\ i}$  need not be particularly special in any way, in terms of the life of our universe. Once we construct the picture, we'll be able to calculate how it would behave-- at times later than  $t_i$  and at times earlier than  $t_i$ .  $t_i$  is just where we are starting.

At time,  $t_i$ , we will give our sphere a maximum size. I'm calling it maximum, because I'm thinking of this as filled with particles. It's, therefore, the maximum radius for any particle. It's just the radius of the sphere.

So  $R_{sub\ max\ i}$  is just the initial radius. An initial means at time,  $t_{sub\ i}$ . We're going to fill the sphere with matter. And we'll think of this matter as being a kind of a uniform fluid, or a dust of very small particles, which we can also think of as a fluid. And it will have a mass density,  $\rho_{sub\ i}$ .

OK. So it's already homogeneous and isotropic, at least isotropic about the center. And now, we want to incorporate Hubble's law. So we're going to start all of this matter, in our toy system here, expanding and expanding in precisely the pattern that Hubble's law requires-- namely, all velocities will be moving out from the center with a magnitude proportional to the distance.

So if I label a particle by-- I'm sorry,  $v_{sub\ i}$  is just for initial. For any particle-- there's no way we're indicating a particle-- at the initial time, the velocity will just obey Hubble's law. It will be equal to some constant, which I'll call  $H_{sub\ i}$ -- the initial value of the Hubble expansion rate-- times the vector  $r$ , which is just a vector from the origin to the particle. That tells us where the particle is that we're talking about.

So  $v_{sub\ i}$  is the initial velocity of any particle.  $H_{sub\ i}$  is the initial Hubble expansion rate. And  $r$  is the position of the particle.

OK. As I said, we're starting with a finite system, which is completely under control. We know, unambiguously, how to calculate-- in principle at least-- how that system will evolve, once we set up these initial conditions. And at the end of the calculation, we'll take the limit as  $R_{max\ i}$  goes to infinity. And that, we hope, will capture the idea that this model universe is going to fill the infinite space.

Now, I did want to say a little bit here-- only because it's something which has entered my scientific life recently and in interesting ways. I would say a few things about infinities.

Now, this is an aside, which means that if you're struggling to understand what's going in the course, you can ignore this and don't worry about it. But if you're interested in thinking about these concepts, the concept of infinity has sort of struck cosmology in the nose in the context of the multiverse, which I spoke of a little bit about in the overview lecture, and which we'll get back to at the end of the course.

The multiverse has forced us to think much more about infinities than we had previously. And in the course of that, I learned some things about infinity that in fact surprised me. For the most part, in physics, we think of infinities as the limit of finite things, as we're doing over here.

So if we want to discuss the behavior of an infinite space, we very frequently in physics start by discussing a finite space, where things are much easier to control mathematically. And then we take the limit as the space gets bigger and bigger.

That works for almost everything we do in physics. And I would say that the reason why it works is because we are assuming fundamentally that physical interactions are local. Things that are vastly faraway don't affect what happens here.

So as we make this sphere larger and larger, we'll be adding matter at larger and larger radii. That new matter that we're adding is not going to have much affect on



what happens inside. And in fact, for this problem, we'll soon see that the extra matter we had on the outside has no effect whatever what happens inside, related to this fact that the gravitational field inside a shell of matter is 0.

So that's a typical situation and gives physicist a very strong motivation to always try to think of infinities as the limit of finite systems. What I want to point out, however, is that that's not always the right thing to do. And there are cases where it's, in fact, emphatically the wrong thing to do. Mathematicians know about this, but physicists tend not to.

So I want to point out that not all infinities are well-described as limits of finite systems. I don't want to call this into question. This, I think, is absolutely solid. And we will continue with it after I go off on this side discussion.

But to give you an example of a system which is infinite and not well-described as a limit of finite systems-- this is a mathematical example-- we could just think of the set of positive integers, also called the natural numbers. And I'll use the letter that's often used for it,  $\mathbb{N}$  with an extra line through it to make it super bold or something like that. So that's the set of integers.

And the question is, suppose we consider trying to describe the set of positive integers, as a limit-- a finite set. So we could think about the limit from  $\mathbb{N}$  goes to infinity-- as  $N$  goes to infinity-- of the natural numbers up to  $N$ . So we're taking sets of integers and taking bigger and bigger sets and trying to take the limit, and asking, does that give us the set of integers?

One might think the answer is yes. What I will claim is that this is definitely not equal to the set of integers. And in fact, I'll claim that the limit does not exist at all, which is why it couldn't possibly be equal to the set integers.

And to drive this home, I just need to remind us what a limit means. And since this is not a math course, I won't give a rigorous definition. But I'll just give you an example that will strike bells and things that you learned in math classes.

Suppose we want to talk about the limit as  $x$  goes to 0 of  $\sin x$  over  $x$ . But we all

know how to do that. It's usually called L'Hopital's rule or something. But you could probably just use the definition of a limit and get it directly. That limit is 1.

And what we mean when we say that, is that as  $x$  gets closer and closer to 0, we could evaluate this expression for any value of  $x$  not equal to 0-- for 0, itself, it's ambiguous. But for any value of  $x$  not equal to 0, we can evaluate this. And as  $x$  gets closer and closer to 0, that evaluation gives us numbers that are closer and closer to 1.

And we can get as close to 1 as we want by choosing  $x$ 's as close to 0 as necessary. And that's usually phrased in terms of epsilons and deltas. But I don't think I need that here. The point is that, the limit is simply the statement that this can be made as close to 1 as you like by choosing  $x$  as close as possible to the limit point, 0.

Now, if you imagine applying the same concept to this set-- the set of integers from 1 to  $N$ -- the question is, as you make  $N$  large, does it get closer to the set of all integers? Are the numbers from 1 to 10 close to the set of all integers? No. What about 1 to a million? Still infinitely far away. 1 to a billion? 1 to a google?

No matter what number you pick as that upper limit, you're still infinitely far away from the set of all integers. You're not coming close. So there's no sense in which this converges to the set of all integers. It's just a different animal.

Now does this make a difference? Are there any questions where it matters whether you think of the integers as being defined in some other way, or by this limit? Or maybe first say how it is defined.

If you ask mathematicians how you define the set of integers, I think all of them will tell you, well, we used the Peano axioms. And the key thing in the Peano axioms-- if you look at them-- that controls the fact that there's an infinite number of integers, is the successor axiom.

That is, built into these Peano axioms that describe the integers mathematically is a statement that every integer has a successor. And then, there are other statements

that guarantee that the successor is not one of the ones that's already on the list. So you always have a new element as the successor to your highest element so far.

And that guarantees from the beginning, the set of axioms is infinite. It's not thought of as the limit of finite sets and cannot be thought of as the limit of finite sets. Because, no finite set is, in any way, resembling an infinite set.

So does it matter? Are there questions where we care whether we could describe the integers this way or not? And the only questions I know sound kind of contrived, I'll admit. But I also point out that in mathematics, the word, contrived doesn't cut much ice. If you discover a contradiction in some system, nobody's going to tell you whether you should ignore that contradiction because it's contrived. If it's really a contradiction, it counts.

So a question where it makes a difference, whether you think of the integers as being defined as infinite from the start, or whether you think of it-- or try to think of it-- as a limit of this sort, is a question such as, what fraction of the integers are so large that they cannot be doubled and still be an integer?

And notice that, if we consider this set, for any  $N$ , no matter how large  $N$  is, we would conclude that half of the integers are so large that they cannot be doubled. And that would hold no matter how large we made  $N$ .

On the other hand, if we look at the actual set of integers, we know that any integer can be doubled and you just get another integer. So that's an example of a property of the integers, which you would get wrong if you thought that you could think of the integers as this kind of a limit. So you really just can't.

Any questions about that? Subtle point, I think.

**AUDIENCE:** How does that relate to--

**PROFESSOR:** To this? It's just a warning that you should be careful about treating infinity as limits of finite things. That's all it is. It does not relate to this.

That's why I said it was an aside when I started. Aside means something worth knowing but not directly related to what we're talking about. OK.

So we're going to proceed with this model. Let me mention one other feature of the model to discuss a little bit, the shape that we started with. We're starting with a sphere. You might ask, why a sphere?

Well, a sphere is certainly the simplest thing we could start with. And a sphere also guarantees isotropy-- or at least isotropy about the origin. Isotropy about one point.

We could, by doing significantly more work, have started, for example, with a cube and let the cube get bigger and bigger. And as the cube got bigger and bigger, it would also fill all space. And we might think that would be another way of getting the same answer. And that would be right.

If we did it with a cube, it would be a lot more work. But we would, in fact, get the same answer. The cube has enough symmetry, So it, in fact, will be the same as sphere in this case.

I'm not going to try to tell you how to calculate it with an arbitrary shape. But I will guarantee you that a cube will get you the same answer.

On the other hand, if we started with a rectangular solid, where the three sides were different-- or at least not all equal to each other-- then would be starting with something which is asymmetric in the first place. Some direction would be privileged.

And then, if we continued-- as we're going to be doing for our sphere-- starting with this irregular rectangular solid, we would build in an anisotropy from the very beginning. We end up with an anisotropic model of the universe.

So since we're trying to model the real universe, which is to a high degree isotropic, we will start with something which guarantees the isotropy. And the sphere does that. And it's the simplest shape which does that.

OK. Now, we're ready to start putting in some dynamics into this model. And the

dynamics that we're going to put in for now will just be purely Newtonian dynamics. And in fact, we'll be modeling the matter that makes up this sphere as just a dust of Newtonian particles-- or if you like, a gas of Newtonian particles.

These particles will be nonrelativistic-- as implied by the word Newtonian. And that describes our real universe for a good chunk of its evolution but not for all of it. So let me-- before we proceed-- say a few words about the real universe and the kind of matter that has dominated it during different eras of evolution.

In the earliest time, our universe, we believe, was radiation dominated. And that means that, if you follow the evolution of our universe backwards in time, as one goes to earlier and earlier times, the photons that make up the cosmic background radiation blue shift.

We've learned that they red shift as universe expands. That means if we extrapolate backwards, they blue shift. They get more energetic for every photon.

But the number of photons remains constant, essentially, as one goes backwards in time. They just get squeezed into a small volume. And they become more energetic.

Now, meanwhile, the particles of ordinary matter-- and dark matter as well, the protons and whatever the dark matter is made of-- also gets squeezed, as you go backwards in time. But they don't become more energetic. They remain just-- a proton remains a particle whose mass is the mass of a proton times  $c$  squared.

So as you go backwards, the energy density in the radiation-- in the cosmic microwave background radiation-- gets to be larger and larger compared to the energy density in the matter. And later, we'll learn how to calculate this. But the two cross at about 50,000 years. Yes.

**AUDIENCE:** If we think of particles as waves, then how does that make sense that they wouldn't change?

**PROFESSOR:** Right. If we think of particles like waves, how does it make sense that they don't change? The answer is, they do a little bit. But we're going to be assuming that

these particles have negligible velocities.

What happens is their momentum actually does blue shift. But a blue shift is proportional to the initial value. And if the initial value is very small, even when it blue shifts, the momentum could still be negligible.

So for the real universe, between time, 0, and about 50,000 years, the universe was radiation dominated. And we'll be talking about that in a few weeks. But today, we're just ignoring that.

Then, from this time of about 50,000 years until about 9 billion years-- so a good chunk of the history of the universe-- the universe was what is called matter dominated. And matter means nonrelativistic matter. That's standard jargon in cosmology.

When we talk about a matter-dominated universe, and even though we don't use the word nonrelativistic, that's what everybody means by a matter dominated universe. And that's the case we're going to be discussing, just ordinary nonrelativistic matter filling space.

And then for our real universe, something else happened at about 9 billion years in its history up to the present-- and presumably in the future as well-- is the universe became dark energy dominated. And dark energy is the stuff that causes the universe to accelerate. So the universe has been accelerating since about 9 billion years from the Big Bang.

Now, I think we mentioned-- I'll mention it again quickly-- there's no conversion of ordinary matter to dark energy here, which you might guess from this change in domination. It's just a question of how they behave as the universe expands. Ordinary matter has a density, which decreases as 1 over the cube of the scale factor. You just have a fixed number of particles scattering out over a larger and larger volume.

The dark energy-- for reasons that we will come to near the end of the course, in fact, does not change its energy density at all as the universe expands. So what

happened 9 billion years ago was simply that the density of ordinary matter fell below the density of dark energy. And then, the dark energy started to dominate and the universe started to accelerate.

Today, the dark energy, by the way, is about 60% or 70% of the total. So it doesn't dominate completely. But it's the largest component.

OK. For today's calculation, we're going to be focusing on this middle period and pretending it's the whole story. And we will come back and discuss these other eras. We're not going to ignore them. But we will not discuss them today.

So we will be discussing this case of a matter-dominated universe. And we're going to be discussing it using Newtonian mechanics. And in spite of the fact that we're using Newtonian mechanics, I will assure you-- and try to give you some arguments later-- it gives exactly the same answer that general relativity gives.

OK. So how do we proceed? How do we write down the equations that describe how this sphere is going to evolve?

We're going to be using a shell picture. That is, we will describe that sphere in terms of shells that make it up. So in other words, we will divide the matter at the initial time. And then, we'll just imagine labeling those particles and following them.

So at the initial time, we're going to divide the matter into concentric shells. So each shell extends from some value of  $r_{sub\ i}$ -- the initial radius-- to  $r_{sub\ i} + dr_{sub\ i}$ . Our shells will be infinitesimal in their thickness.

And each shell will have a different  $r_{sub\ i}$ . And we could let all the  $dr_{sub\ i}$ 's be the same, if we want. We can let each shell be the same thickness. We can let them be different. It doesn't matter.

Now a reason why we can get away with thinking of the matter purely as being described by shells is that we know that we started with all of the velocities radial. That Hubble's law that we assumed says that the velocities were proportional to the radius vector, which is measured from the origin. So all our initial velocities were

radial outward.

And furthermore, if we think about how the Newtonian force of gravity will come about for this spherical object, they will all be radial as well. So all the motion will remain radial. There will never be any forces acting on any of our particles that will push them in a tangential direction.

So all motion will be radial. And as long as we keep track of the radius of each particle, its angular variables, theta and phi-- which I will never mention again-- will just be constant in time. So we will not need to mention them.

So all motion is radial, because the v's start radial. And there are no tangential forces, where tangential means any direction other than radial. So we're just saying that all the forces are radial. And that means the motion will stay radial.

To describe the motion, I want to give a little bit more teeth to this statement that we're going to be describing it in terms of shells. We'll going to describe the motion by a function,  $r$ , which will be a function of two variables,  $r$  sub  $i$  and  $t$ .

And this is just the radius at time,  $t$ , of the shell that was at radius,  $r$  sub  $i$ , at time,  $t$  sub  $i$ . So each shell is labeled by where it is at time,  $t$  sub  $i$ . And once we label it, it keeps that label.

But then, we'll talk about this function,  $r$ , which tells us where it is at any later time. And we could also think about earlier times if we want. So  $r$  of  $r$  sub  $i$ ,  $t$  is the radius of the shell that started at  $r$  sub  $i$ , where we're looking at the radius at time,  $t$ .

Now, I should warn you that if you look in any textbook, you will see a simpler derivation than what I'm about to show you. So you might wonder why am I going to so much trouble when there's an easier way to do it. And the answer is that the calculation I show you will, in fact, show you more than what's in the textbooks-- and Ryden's textbook, for example.

What most textbooks do-- including Ryden, I believe-- is to assume that this motion will continue to obey Hubble's law and continue to be completely uniform in the



density. And then, you could just calculate what happens to the outside of the sphere. And that governs everything, if you assume that will stay uniform.

We are not going to assume that it will stay uniform. We will show that it will stay uniform. And from my point of view, it's a lot better to actually show something than to just guess it and we write about, without showing it.

So that's why we're going to do some extra work. We will actually show that the uniformity, once you put in, is preserved under time, under the laws of Newtonian evolution.

I guess I'll leave the picture. Maybe, I'll leave all of that.

OK. Now, there's an issue that's a little bit complicated that I'll try to describe. And again, this is a subtlety that's probably not mentioned in the books.

We have these different shells that are evolving. And we can calculate the force on any given shell if we know the matter that's inside that shell. That's what it tells us. If everything is in shells, the shells have no effect on the matter that is inside the shell. So if we want to know the effect on a given shell, we only need to know the matter that's inside that shell. The matter outside has no force.

So it's very important that we know the ordering of these shells. Now, initially, we certainly do. They're just ordered according to  $r$  sub  $i$ . But once they start to move around, there's, in principle, the possibility that shells could cross.

And if shells crossed, our equations of motion would have to change, because then different amounts of mass would be acting on different shells. And we'd have to take that into account.

Fortunately, that turns out not to be a problem. It just sounds like it could be a problem. And the way we're going to treat it is to recognize that, initially, all these shells are moving away from each other, because of the Hubble expansion.

If Hubble's law holds, any two particles are moving away from each other with a relative velocity proportional to their distance. So that holds for any two shells, as

well.

So if shells are going cross, they're certainly not going to cross immediately. There are no two shells that are approaching each other initially. All shells are moving apart initially.

This could be turned around by the forces-- and we'll have to see. But what we can do is, we can write down equations that we know will hold, at least until the time where there might be some shell crossings. That is, we'll write down equations that will hold as long as there are no shell crossings.

Then, if there was going to be a shell crossing, the equations we write down would have to be valid right up until the time of that shell crossing. And, therefore, the equations that we're writing down would have to show us that the shells are going to cross. The shells can't just decide to cross independently of our equations of motion.

And what we'll find is that our equations will tell us that there will be no shell crossings. And the equations are valid as long as there are no shell crossings. And I think if you think about that, that's an airtight argument, even though we're never going to write down equations that will tell us what would happen if shells did cross.

OK. So we're going to write equations that hold as long as there are no shell crossings. OK. As long as there's no shell crossings, then the total mass inside of any shell is independent of time. It's just the shells that are inside it.

So the shell at initial radius,  $r_{sub\ i}$ , even at later times, feels the force of the mass inside. And we can write down a formula for that mass inside. The mass inside the shell, whose initial radius is  $r_{sub\ i}$ , is just equal to the initial volume of that sphere, which is  $\frac{4\pi}{3} r_{sub\ i}^3$ , times the initial mass density,  $\rho_{sub\ i}$ .

So that's how much mass there is inside a given shell when the system starts. And that will continue to be exactly how much mass is inside that shell as the system evolves, unless there are shell crossings. And our goal is simply to write down

equations that are valid until there might be a shell crossing. And then, we can ask whether or not there will be any shell crossings.

OK, now, Newton's law tells us how to write down the acceleration of an arbitrary particle in this system. Newton's law tells us the acceleration will be proportional to negative-- I'll put a radial vector at the end in  $\hat{r}$ . So its direction, minus  $\hat{r}$ , will be the direction enforced on any particle.

And the magnitude, it will be Newton's constant times the mass enclosed in the sphere of radius,  $r$  sub  $i$ , divided by the square of the distance of that shell from the origin. And that's exactly what we called this function,  $r$  of  $r_i$ ,  $t$ . It's the radius of the shell at any given time.

So it's  $r$  squared of  $r$  sub  $i$  and  $t$ . And then as I promised you, there's a unit vector,  $\hat{r}$ , at the end. So the force is-- and the acceleration, therefore,-- is in the negative  $\hat{r}$  direction.

So this holds for any shell, whether which shell we're talking about is indicated by this variable,  $r$  sub  $i$ .  $r$  sub  $i$  tells us the initial position of that shell.

So is everybody happy with this equation? This is really the crucial thing. Once you write this down, everything else is really just chugging along. So everybody happy with that?

It's just the statement from Newton, that if all the masses are ranged spherically symmetrically, the mass inside any shell-- excuse me, the force due to the mass that's on a shell that's at a larger radius than you are, produces no acceleration for you. The only acceleration you feel is due to the masses at smaller radii. And that's what this formula says. OK, good.

Now, what I want to do is-- this is a vector equation-- but we know that we're just taking these spherical motions. And all we really have to do is keep track of how  $r$  changes with time, little  $r$ .

So we can turn this into an ordinary differential equation with no vectors for this

function, little  $r$ . The acceleration has a magnitude, which is just,  $r$  double dot. And that is then equal to minus-- and we write again what  $M$  of  $r_i$  is from this formula. So I'm going to write this as minus  $4\pi$  over  $3G$  times  $r$  sub  $i$  cubed  $\rho$  sub  $i$ -- coming from that formula-- divided by  $r$  squared.

And I'm going to stop writing the arguments. But this  $r$  is the function of  $r$  sub  $i$  and  $t$ . But I will stop writing its arguments. And the two dots means derivative with respect to time. Yes.

**AUDIENCE:** I guess I don't understand how come we would, calculating the mass, we use  $r$  sub  $i$ , which is the radius at some initial time,  $t_i$ . But then, when we're saying the distance from that mass to our point, we use this function,  $r$ , instead of  $r$  sub  $i$ .

**PROFESSOR:** Right. An important question. The reason is that we're using the initial radius, but we're also using the initial density. So this formula certainly gives us the mass that's inside that circle at the beginning. And then, if there's no shell crossings, all these circles move together. The mass inside never changes. So it's the same as it was then.

While, the distance from the center of that mass does change, as the radius changes. So the denominator changes, the numerator does not. Yes.

**AUDIENCE:** So, the first equation,  $g$ , that's the gravity--

**PROFESSOR:**  $g$  is the acceleration of gravity.

**AUDIENCE:** So, why does it have-- it has units of-- no, it doesn't. Sorry.

**PROFESSOR:** Yes, it should have units of acceleration, as our double dot should have units of acceleration. Hopefully, they do.

OK. Now, as the system evolves,  $r$  sub  $i$  is just a constant-- different for each shell, but a constant in time. And imagine solving this one shell at a time.  $\rho$  sub  $i$  is, again, a constant. It's the value of the mass density at the initial time. So it's going to keep its value.

So the differential equation involves a differential equation in which little  $r$  changes with time, and nothing else here does. So this is just a second order differential equation for little  $r$ . So, we're well on the way to having a solution, because second order differential equations are, in principle, solvable.

But the one thing that you should all remember about second order differential equations is that in order to have a unique solution, you need initial conditions. And if it's a second order equation-- as Newton's equations always are-- you need to be able to specify the initial position and the initial velocity before the second order equation leads to a unique answer.

So that's what we want to do next. We want to write down the initial value of  $r$  and the initial value of  $r$  dot. And then, we'll have a system, which is just something we can turn over to a mathematician. And if the mathematician is at all smart, he'll be able to solve it. So initial conditions.

So we want the initial value of  $r$  sub  $i$ -- and initial means a time,  $t$  sub  $i$ . That's where we're setting our initial conditions. So we need to know what  $r$  of  $r$  sub  $i$ ,  $t$  sub  $i$  is, which is a trivial thing if you keep track of what this notation means. What is that?  $r$  sub  $i$ , good.

The meaning of this is just the radius at time,  $t$  sub  $i$ , of the particle, whose radius at time,  $t$  sub  $i$ , was  $r$  sub  $i$ . So to put together all those tautologies, the answer is just, obviously,  $r$  sub  $i$ . OK. And then we also want to be able to solve this equation and have a unique solution-- we want the initial value of  $r$  sub  $i$  dot. And initial means, again, at time,  $t$  sub  $i$ . And that's specified by our Hubble expansion. Every initial velocity is just  $H$  sub  $i$  times the radius.

So if the radius is  $r$  sub  $i$ , this is just  $H$  sub  $i$  times  $r$  sub  $i$ . That's just the Hubble velocity that we put in to the system when we started it.

OK. So now, we have a system, which is purely mathematical. We have a second order differential equation and initial conditions on  $r$  and  $r$  dot. That leads to a unique solution. Now it's pure math. No more physics, at least at this stage of the

game.

However, there are interesting things mathematically that one can notice about this system of equations. And now what we're going to see is the magic of these equations for preserving the uniformity of this system. It's all built into these equations. And let's see how.

The key feature that's somewhat miraculous that these equations have is that  $r_{sub\ i}$  can be made to disappear by a change of variables. I'll tell you what that is in a second. It also helps if you know how to spell "disappear."

We're going to define a new function,  $u$ . I just made up a letter there. Could have called it anything. But  $u$  of  $r_{sub\ i}$  and  $t$  is just going to be defined to be  $r$  of  $r_{sub\ i}$  and  $t$  divided by  $r_{sub\ i}$ . Given any function,  $r$  of  $r_{sub\ i}$  and  $t$ , I can always define a new function, which is just the same function divided by  $r_{sub\ i}$ .

Now, let's look at what this does to our equations. My claim is that  $r_{sub\ i}$  was going to disappear. And now, we'll see how that happens.

OK, if  $u$  is defined this way, we can write down the differential equation that it will obey, by writing down an equation for  $u$  double dot. And  $u$  double dot will just be  $r$  double dot divided by  $r_{sub\ i}$ . So we take the equation for  $r$  double dot over here and divide it by  $r_{sub\ i}$ .

So we can write that as  $-\frac{4\pi}{3}G$ -- for some reason, I decided to leave the  $r_{sub\ i}$  cubed in the numerator for now-- and just put an extra  $r_{sub\ i}$  in the denominator-- the one that we divided by-- times  $r$  squared.

And now, what I'm going to do is just replace  $r$ . We're trying to write an equation for  $u$ .  $r$  is related to  $u$  by this equation.  $r$  is  $r_{sub\ i}$  times  $u$ .

So I can write this as  $-\frac{4\pi}{3}G r_{sub\ i}$  cubed  $\rho_{sub\ i}$  over-- now we have--  $u$  squared times  $r_{sub\ i}$  cubed. I left the numerator alone. I just rewrote the denominator by placing  $r$  by  $r_{sub\ i}$  times  $u$ . And we get this formula.

And now, of course, the  $r_{sub\ i}$  cubes cancel. And I think the reason why I kept it

separated this way when I wrote my notes is at this shows very explicitly that what you have is a cancellation between an  $r$  sub  $i$  here, which was the power of  $r$  that appears in a volume--  $r$  sub  $i$  cubed was just proportional to the volume of the sphere-- and  $r$  sub  $i$  cubed down here, where one of them came from just a change of variables, and the other was the  $r$  squared that appeared in the force law.

So this cancellation depends crucially on the force law, being a  $1$  over  $r$  squared force law. If we had  $1$  over some other power of  $r$ -- even if it differed by just a little bit-- then the  $r$  sub  $i$  would not drop out of this formula. And we'll see in a minute that it's the dropping out of  $r$  sub  $i$  which is crucial to the maintenance of homogeneity as the system evolves.

As it evolves, Newton tells us what happens. We don't get to make any further choices. And if Newton is telling us there's a  $1$  over  $r$  squared force law, then it remains homogeneous. But otherwise, it would not. And that's, I think, a very interesting fact.

Continuing, we've now crossed off these  $r$  sub  $i$ 's. And we get now a simple equation,  $u$  double dot is equal to minus  $4\pi$  over  $3G\rho$  sub  $i$  over  $u$  squared with no  $r$  sub  $i$ 's in the equation anymore. And that means that this  $u$  tells us the solution for every  $r$  sub  $i$ .

We don't have different solutions for every value of  $r$  sub  $i$  anymore.  $r$  sub  $i$  has dropped out of the problem. We have a unique solution independent of  $r$  sub  $i$ . And that means it holds for all  $r$  sub  $i$ .

I'm sorry. I jumped the gun. The conclusions I just told you about are true, but part of the logic I haven't done yet.

What part of the logic did I leave out? Initial conditions, I heard somebody mumble. Yes. Exactly.

To get the same solution, we have to not only have a differential equation that's independent of  $u$  sub  $i$ , but we don't have a unique solution unless we look at the

initial conditions. We better see if the initial conditions depend on  $u$  sub  $i$ . And they don't either. That's the beauty of it all.

Starting with the  $r$  initial condition, the initial value of  $u$  of  $r$  sub  $i$ ,  $t$  sub  $i$ , will just be equal to the initial value of  $r$  divided by  $r$  sub  $i$ . But the initial value of  $r$  is  $r$  sub  $i$ . So here, we get  $r$  sub  $i$  divided by  $r$  sub  $i$ , which is 1, independent of  $r$  sub  $i$ . The initial value of  $u$  is 1, for any  $r$  sub  $i$ .

And similarly, we now want to look at  $u$  dot of  $r$  sub  $i$  and  $t$ . And that will just be  $r$  dot divided by  $r$  sub  $i$ , where we take the initial value of  $r$  dot. The initial value of  $r$  dot is  $H_i$  times  $r$  sub  $i$ . The  $r$  sub  $i$ 's cancel. And  $u$  sub  $i$  dot is just  $H$  sub  $i$ .

So now, we have justified the claim that I falsely made a little bit prematurely a minute ago. We have a system of equations, which can be solved to give us a solution for  $u$ , which is independent of  $r$  sub  $i$ , which means every value of  $r$  sub  $i$  has the same equation for  $u$ .

That-- if you stare at it a little bit-- gives us a physical interpretation for this quantity,  $u$ .  $u$ , in fact, is nothing more nor less than the scale factor that we spoke about earlier. We have found that we do have a homogeneously expanding system.

We started it homogeneously expanding, but we didn't know until we looked at equation of motion that it would continue to homogeneously expand. But it does. And that means it can be described by a scale factor.

We have  $u$  of  $r$  sub  $i$ ,  $t$ . In principle, it was defined so it might have depended on  $r$  sub  $i$ . That's why I've been writing these  $r$  sub  $i$ 's all along.

But now, we discovered that the  $u$ 's are completely determined by these equations, which have no  $r$  sub  $i$ 's in them-- at least none that survived. We had an  $r$  sub  $i$  divided by  $r$  sub  $i$ , but that's not really an  $r$  sub  $i$ , as you know. It's 1.

So  $u$  is independent of  $r$  sub  $i$  and can be thought of as just a function of  $t$ . And we can also change its name to  $a$  of  $t$  to make contact with the notion of a scale factor that we discussed last time. And we can see that the way  $u$  arises, in terms of how it



describes the motion,  $r$  of  $r$  sub  $i$  and  $t$  is just equal to-- from this equation--  $r$  sub  $i$  times  $u$ , which is now called  $a$ .

So  $r$ , of  $r$  sub  $i$  and  $t$ , is equal to  $a$  of  $t$  times  $r$  sub  $i$ . That's just another way of rewriting our definition that we started with of  $u$ .

So what does that mean? These  $r$  sub  $i$ 's are comoving coordinates. We've labeled each shell by its initial position,  $r$  sub  $i$ . And we've let that shell keep its label,  $r$  sub  $i$ , as it evolved. That's a comoving coordinate. It labels the particles independent of where they move.

And what this is telling us is that the physical distance-- from the origin in this case-- is equal to the scale factor times the coordinate distance. That is the coordinate distance between the shell and the center of the system, the origin.

OK. Any questions about that?

OK. It's now useful to rewrite these equations in a few different ways.

Let's see. First of all, we have written our differential equation, up here, in terms of  $\rho$  sub  $i$ . And that was very useful because  $\rho$  sub  $i$  is a constant. It doesn't change with time.

Still, it's sometimes useful to write the differential equation in terms of the temporary value of  $\rho$ , which changes with time, to see what the relationships are between physical quantities at a given time. And we could certainly do that, because we know what the density will be at any given time.

For any shell, we can calculate the density as the total mass divided by the volume. We know this is going to stay uniform, because everything is moving together, when you just have motion, which is an overall scale factor that multiplies all distances. So the density will be constant.

And we can calculate the density inside a shell by taking  $M$  of  $r$  sub  $i$ -- which we already have formulas for and is independent of time-- and dividing it by the volume inside the shell, which is  $4\pi r$  cubed. And doing some substitutions,  $M$  of  $r$  sub  $i$  is

just the initial volume times the initial density. So that is-- as we've written before--  $4\pi/3 r_i^3 \rho_i$ .

And then, we can write the denominator.  $r$  is equal to  $a$  times  $r_i$ . The physical radius is the scale factor times the coordinate radius. So here, we have  $a^3$  times  $r_i^3$ . And now, notice that almost everything cancels. And what we're left with is just  $\rho_i$  divided by  $a^3$ .

So that's certainly what we would've guess, I think. The density is just what it started but then divided by the cube of the scale factor. And our scale factor is defined as 1 at the initial time-- the way we've set up these conventions. So that is just the ratio of the scale factors cubed that appears in that equation. As the universe expands, the density falls off by  $1/a^3$ .

We can also now rewrite the equation for  $\ddot{a}$ .  $a$  is  $u$ , so we have the equation up there. But we could write it in terms of the current mass density. Starting with what we have up there, it's  $-\frac{4\pi}{3} G \rho_i / a^2$ .

Notice, that I can multiply numerator and denominator by  $a$ . And then, we have  $\rho_i / a^3$  here, which is just the mass density in any given time. So I can make that substitution. And I get the meaningful equation that  $\ddot{a}$  is equal to  $-\frac{4\pi}{3} G \rho(t) a$ .

So this equation gives the deceleration of our model universe in terms of the current mass density. And notice that it does, in fact, depend only on the current mass density. It predicts the ratio of  $\ddot{a}$  over  $a$ .

And that, you would expect to be what should be predicted, because remember,  $a$  is still measured in notches per meter. So the only way the notches can cancel out is if there is an  $a$  on both sides. Or if you could bring them all to one side and have  $\ddot{a}$  divided by  $a$ . And then, the notches go away. And you have something which has physical units being related to something with physical units.

OK. We said at the beginning that when we were finished, we were going to take the limit as  $R \rightarrow \infty$ . And lots of times when I present this, I forget to talk about that.

And the reason I forget to talk about that is, if you notice,  $R$  doesn't appear in any of these equations. So taking the limit as  $R \rightarrow \infty$  doesn't actually involve doing anything. It really just involves pointing out that the answers we got are independent of how big the sphere is, as long as everything we want to talk about fits inside the sphere.

Adding extra matter on the outside doesn't change anything at all. So taking the limit as you add an infinite amount of matter on the outside-- as long as you imagine doing it in these spherical shells-- is a trivial matter. So the limit as  $R \rightarrow \infty$  is done without any work.

OK. I would like to go ahead now, and in the end, we're going to want to think about different kinds of solutions to this equation and what they look like. For today, I want to take one more step, which is to rewrite this equation in a slightly different way, which will help us to see what the solutions look like. What I want to do is to find a first integral of this equation.

OK. To find a first integral, I'm going to go back to the form that we had on the top there, where everything is expressed in terms of  $\rho$  rather than  $\rho$ . And the advantage of that for current purposes is that I really want to look at the time dependence of things. And  $\rho$  has its own time dependence, which I don't want to worry about. So if I look at the formula in terms of  $\rho$ , all time dependence is explicit.

So I'm going to write the differential equation. It's the top equation in the box there, but I'm replacing  $u$  by  $a$ , because we renamed  $u$  as  $a$ . And I'm going to put everything on one side. So I'm going to write  $\ddot{a} + \frac{4\pi}{3} G \rho = 0$ . OK, that's our differential equation.

Now, it's a second order differential equation, like we're very much accustomed to

from Newtonian mechanics-- as this is an equation which determines a double dot, the acceleration of  $a$  in terms of the value of  $a$ . A common thing to make use of in Newtonian mechanics is conservation of energy.

In this case, I don't know if we should call this conservation of energy or not. We'll talk later about what physical significance the quantities that we're dealing with have. But certainly, as a mathematical technique, we can do the same thing that would have been done if this were a Newtonian mechanics problem, and somebody asked you to derive the conserved energy.

Now, you might have forgotten how to do that. But I'll remind you. To get the conserved energy that goes with this equation, you put brackets around it. You could choose whether you want curly brackets or square brackets or just ordinary parentheses.

But then, the important thing is that it's useful to multiply the entire equation by an integrating factor, a dot. And once you do that, this entire expression is a total derivative. This equation is equivalent to  $dE$ -- for some  $E$  that I'll define in a second--  $- dt$  equals zero, where  $E$  equals  $1/2 \dot{a}^2$  minus  $4\pi/3 G \rho$  sub  $i$  over  $a$ .

And you can easily check. If I differentiate this, I get exactly that equation. So they're equivalent. That is, this is equivalent to that. So  $E$  is a conserved quantity.

Now if one wants to relate this to the energy of something, there are different ways you can do it. One way to do it is to multiply by an expression, which I'll write down in a second, and to think of it as the energy of a test particle at the surface of this sphere. I'll show you how that works.

I'm going to find something, which I'll call  $E_{\text{phys}}$ -- or physical-- meaning, it's the physical energy of this hypothetical test particle. It's not all that physical. But it will just be defined to be  $m r_{\text{sub } i}^2$  times  $E$ .  $m$  is the mass of my test particle.  $r_{\text{sub } i}$  is the radius of that test particle expressed in terms of its initial value.

And then, if we write down what  $E_{\text{phys}}$  looks like, absorbing these factors, it can be

written as  $\frac{1}{2} m \dot{r}_i^2 - \frac{GmM}{r_i}$ . And that's just some algebra-- absorbed these extra factors that I put into the definition.

And now, if we think of this as describing a test particle, where  $r_i$  is  $R_{\text{max}}$ -- so we're talking about the boundary of our sphere-- then, we can identify what's being conserved here. What's being conserved here is  $\frac{1}{2} m v^2 - \frac{GmM}{R_{\text{max}}}$  would just be the velocity of the particle of the boundary of the sphere. And then, minus  $G$  times the product of the masses divided by the distance between the particle and the center.

And that would just be the Newtonian energy-- kinetic energy plus potential energy, where the potential energy is negative-- of a point particle on the boundary, where we let  $r_i$  be  $R_{\text{max}}$ -- the boundary of the sphere.

Now, if we want to apply this to a particle inside the sphere, it's a little trickier to get the words right. If the particle is inside the sphere-- if  $r_i$  is not equal to the max-- this is not really the potential energy of the particle. Can somebody tell me why not?

Well, maybe the question is a little vague. But if I did want to calculate the potential energy of a particle inside the sphere-- that's meant to be in the interior. You can't really tell unless it's the actual diagram. But that's deep inside sphere.

I would do it by integrating from infinity  $G da$ , and ask how much work do I have to do to bring in a particle from infinity and put it there. And in doing this line integral, I get a contribution from the mass that's inside this dot, which is what determines the force on that dot. But I also get a contribution from what's outside the dot.

So I don't get-- if I wanted to calculate the actual potential energy of that point-- I don't get simply  $Gm$  times the mass inside divided by the distance from the center. It's more complicated what I get.

And in fact, what I get is not conserved. Why is it not conserved? I could ask you, but I'll tell you. We're running out of time.

It's not conserved, because if you ask for the potential energy of something in the presence of moving masses, there's no reason for it to be conserved. The potential energy for a point particle moving in the field of static masses is conserved. That's what you've learned in [? AO1 ?] or whatever.

But if other particles are moving, the total energy of the full system will be conserved. But the potential energy of a single particle-- just thought of as a particle moving in the potential of the other particles-- will not be conserved.

OK. What is conserved, besides the energy of this test particle on the boundary, is the total energy of this system, which can also be calculated. And that, in fact, will be one of your homework problems for the coming problem set-- to calculate the total energy of that sphere. And that will be related to this quantity with a different constant of proportionality and will be conserved for the obvious physical reason.

For the particles inside, what one can imagine-- and I'll just say this and then I'll stop-- you can imagine that you know that the motion of this particle is uninfluenced by these particles outside. And therefore, you could pretend that they're not there and think of it as an analog problem, where the particles outside of the radius of the particle you're focusing on simply do not exist in this analog problem.

For that analog problem, this would be the potential energy. And it would be conserved. And you could argue that way, that you expect this to be a constant. And you'd be correct. But it's a little subtle to understand exactly what's conserved and why and how to use it.

OK, that's all for today.