PROFESSOR: OK, in that case, let's get started. As usual, I like to begin by giving a review of what we talked about last time. This time on slides instead of on the blackboard. We're talking mainly about relativistic energy, or relativistic energy and momentum, and pressure, sometimes.

The key equation is probably the most famous equation in physics, Einstein's e equals m c squared. And I gave some numerical examples. I actually looked up some more numbers since then, when I was revising the lecture notes. So these are slightly more up to date. But it's still true that if you could burn matter at the rate of one kilogram per hour, you would have about one and a half times the total power output of the world.

And that's apparently still valid in 2011. I only had 2008 figures, actually, at the lecture last time. And if you imagine the 15-gallon tank of gasoline, and you could figure out how much that-- what it's mass is, and convert that to energy, it turns out that a 15-gallon tank of gasoline could power the world for about two and half days, if you could convert all of it into energy.

The catch, of course, is that we can't convert matter into energy. We can't get around the problem that, at least at the energies that we deal with, baryon number. And that number of protons and neutrons is conserved, so we can't make protons and neutrons disappear. And that means that we're limited in what we can do.

In particular, one of the most efficient things we can do is fission uranium 235. But when uranium 235 undergoes fission, less than 1/10 of 1% of the mass is actually converted into energy, which is why we can't actually avail ourselves of these fantastic numbers that would apply, if we could literally just convert matter into energy.

We went on to talk about the relativistic definitions of energy and momentum, and how they come together to form a Lorentz four vector, and the underlying theme here is that we consider ourselves users of special relativity. Most of you I know have taken special relativity courses, and for those of you who have, this is a review. For those you who have not, and there are some of those also, no need to panic.

I intend to tell you every fact that you need to know about special relativity. I won't tell you how to derive them all, but I'll tell you all you'll need to know for this class. So in particular, it's useful for this class to recognize that energy and momentum can be put together into a four vector, where the zeroth component is the energy divided by the speed of light. And the three spatial components are just the three components of the spatial momentum, although they have to be defined relativistically.

The relativistic definition of momentum, at least how it relates to velocity, is that it's equal to gamma times the rest mass times the velocity, where gamma is the famous factor that we've been seeing all along when we've talked about relativity. The Lorentz contraction factor, one over the square root of 1 minus v squared over c squared. The energy of a particle, relativistically, is the same gamma, times m 0 times c squared, and it can also be written as the square root of m 0 c squared squared, plus the momentum squared, times c squared.

Since the momentum forms a four vector, its Lorentz and variant square should be Lorentz and variant, and that means that the momentum squared minus p 0 squared should be the same in all inertial reference frames. And that's just equal when you put in what p 0 means, the momentum squared minus the energy squared, divided by c 0 squared.

And to know what value it's equal to in all frames, it's efficient to know what it's equal to in one frame. And the one frame where we do know what it's equal to is the rest frame of a particle. So in the rest frame, the momentum vanishes, and the energy is just m 0 c squared. So in the rest frame, we can evaluate this, and we get minus m 0 c squared squared. And that means that has to be the value in every frame. And this in fact is the easy way to derive the relationship between energy and momentum.

If we go back, the equation we had relating energy and momentum is really exactly that equation, rearranged. Just to give an example of how this works, when we actually have energy exchanges, I pointed out that we could talk about the energy of a hydrogen atom. And because energy and mass are equivalent, the hydrogen atom clearly has a little bit less energy than an isolated proton, plus an isolated electron.

Because when you bring them together there's a binding energy, and that binding energy is called delta e, and has a value of 13.6 electron volts for the ground state of hydrogen. So that tells us the mass of an hydrogen atom is not the sum of the two masses, but rather has this correction factor, because we've taken out a little bit of energy for the binding. And that means we've taken out a little bit of mass.

OK, then we talked about the mass density of radiation and how-- building up to how that will affect the universe. And we said that the mass density of radiation is just the energy density divided by c squared. And that can be taken, really, as a definition of what we call relativistic mass, and hence relativistic mass density.

But the important point is that this mass density actually does gravitate the same as any other mass density of the same value. It really does create gravity in the same way. Now I mentioned that things are much more complicated if you want to talk about the gravitational field produced by a single moving particle.

That's asymmetric, the velocity of the particle shows up in the equations that describe the metric surrounding a single moving particle. But if you have a gas of particles moving at high velocities, where the velocities nonetheless average to zero, which tends to happen, in a gas at least in the rest frame of the gas.

Then that gas will produce gravitational fields, just like a static mass density. Where the mass density is this relativistic energy divided by c squared, the relativistic definition of the mass density. It's also useful to know that the photon-- if we want to describe it as a particle, is a particle of zero rest mass. Which means that it can never be at rest, it always moves at the speed of light.

And it also means that its energy can be arbitrarily small, because the energy is proportional of momentum, and the momentum of a photon can be as small as you like. For giving frequency, of course, the energy of a photon is fixed. It's h times nu but if you're allowed to vary the frequency, which you can do if you just look at it in different frames, you can make the energy as small as you like.

And the famous equation then, p squared minus e squared c squared, which would have on the right, minus m 0 squared c to the 4th m 0 squared c squared, excuse me-- has zero on the right hand side, because m 0 is 0 And that means that for photons, the energy is just the speed of light times the momentum. And that's a famous relationship that photons obey.

Now, thinking about how this gas of photons will behave in the universe, we realized immediately that it does not behave the same way as a mass density of ordinary non-relativistic particles. Which is what we have been dealing with to date. The important difference is that in both cases, the number density falls off like 1 over a cubed, as the universe expands, these particles are not created and destroyed in significant numbers, they just persevere.

So the number density of either non-relativistic particles, or photons, just falls off like one over the volume, as the volume increases and the number density dilutes. But what makes photons different from non-relativistic particles, is that a non-relativistic particle will maintain the energy of that particle as the universe expands, but photons will redshift as the universe expands.

So each photon will itself lose energy. And it loses energy proportional to one over the scale factor. And that's just because the frequency shift proportionally to the scale factor. And that means that the energy per photon shifts, because quantum mechanically we know that the energy of a photon is proportional to its frequency.

So if the frequency redshifts so must the energy. In exactly the same way, 1 over a f t.

Yes, question?

AUDIENCE: You said previously that neutrinos behave like radiation in the sense that theta energy falls is 1 over a. What is it about them that makes this happen? Because there are also particles with standard kinetic energy, right?

PROFESSOR: OK, the question, in case you didn't hear it, is why-- how did neutrinos fit in here? I've made the claim that neutrinos act like radiation in the early universe, but neutrinos have a non-zero mass, so they should obey the standard formulas for particles with nonzero masses. The answer to that is-- there is, I think, a simple answer, which is that as long as the energy is large compared to the mass, particles with masses will still act like massless particles.

It doesn't really matter if the mass is zero or not, the key thing, really, is this equation. So if the term on the right hand side is small compared to either of the two on the left, it's not much different from being zero. And that's what happens for neutrinos in the early universe. And we'll see soon that if you go to early enough times, it's true even for electron-positron pairs. They will also act like radiation. Any particle will act like radiation as long as the energy is large compared to the rest energy.

So getting back to the discussion of the early universe, if the energy of each photon falls off like one over the scale factor, and the number density falls off like one over the scale factor cubed, it means that the energy density, and hence, the mass density of radiation will fall off like one over a to the fourth, in contrast to the one over a cubed, that we found when we were talking about non-relativistic matter.

And that, of course, is going to make a difference. Because those issues play a key role in our discussions about how the universe evolved. An important feature, which we see immediately, is that if we extrapolate backwards in time, since the radiation mass density is falling off like one over a to the fourth, and the matter density is falling off like one over a cubed, it means that as you go back in time, the radiation becomes more and more important relative to the matter, by factor of the scale factor.

So if we go back far enough, we will even find a time when the mass density in radiation equaled the mass density in non-relativistic matter. And we calculated about when that would be. We said that the energy density in radiation today is given by this number, 7 times 10 to the minus 14, joules per meter cubed.

And I just gave you this number, I didn't derive it yet. We will derive it, probably later today. But for now, we're just accepting it. And that implies we can calculate from that the ratio of the mass densities in radiation and ordinary matter. Here, we use the fact that ordinary matter can be described by having an omega ratio to the critical density of about 0.3. And we know how to calculate the critical density, and that allowed us to calculate the density of ordinary matter.

And then this ratio turned out to be 3.1 times 10 to the minus four. So radiation in today's universe is almost negligible in its contribution to the overall energy balance, compared to non-relativistic matter. But if you extrapolate backwards, we know that this ratio will vary as one over the scale factor. And we could figure out what constants to put this equation by putting in the right constant, so that this equation gives us the right value today. Where the right value today is 3.1 times 10 to the minus four.

And notice that this works, if we let t be equal to t sub zero, this factors one and we get 3.1 times 10 to the minus four. So these two factors together they have t zero and the 3.1 times 10 to the minus four, are just the right factors to put in to give us the right constant of proportionality in that equation. Having this equation, we can then ask, how far back do we have to go, how much we have to change t, for the ratio to be one? And that's a straightforward calculation.

And the ratio of the a is then just one over 3.1 times 10 to the minus four, or 3,200. So if we talk about it in terms of a redshift, which is how astronomers always talk about distances, or times, we're talking about going back to a redshift of 3,200. We can figure out what time, then, corresponds to also, if we know how a f t depends on time.

And we do, approximately. For this calculation, I assume for now, we could do better later, and we will-- but I assume for now that we could just treat the period between matter radiation equality, so-called t x and now as being entirely described by a matter-dominated universe. That's only a crude approximation, but it will get us the right order of magnitude.

And we'll learn how to do better later in the course. So if we assume that, then t x is just this number to the 3/2 power, cancelling the 2/3, times the age of universe, t naught. And that turned out to be about 75,000 years. So somewhere in the range of 100,000 years, 50,000 years, is the time in the history of the universe when radiation ceased to be more important than matter.

And for earlier times than that, the radiation dominated. And that's what we refer to as the radiation dominated era. Any questions?

OK, now I think we get on to what is really the important subject that we want to understand, and most of this you did yourself on the homework. But I'll summarize the argument here. We want to understand what this tells us about the Friedmann Equations. And first, we'd like to understand what it says about pressure. Because it turns out that pressure is the crucial issue in determining how fast row falls off as a expands.

So if row is proportional to one over a cubed, we can just differentiate that, putting in a constant proportionality temporarily, just to keep track of what we're doing. Since we know how to differentiate qualities and we're less familiar with how to differentiate proportionalities. But what we find immediately, is that row dot is then minus 3, where that 3 is that 3, times a dot over a times row.

On the other hand, if row of t falls like one over a to the fourth, row dot is minus 4 times a dot over a times row, just by differentiation. So we get different expressions from row dot, between radiation and matter. And we want to explore the consequences of that difference. It's related to the pressure of the gas, because we can relate the pressure to row dot.

Because we know that as a gas expands, it loses energy, which is just equal in amount to pdV. And we illustrated this by a piston thought experiment, but it's true in general. So we can apply this famous formula, dU equals minus pdV, to a patch of the expanding universe. And by a patch I mean some fixed region and coordinate space.

So the total energy in that region of coordinate space will be the physical volume, which will be a cubed times the coordinate volume. Which is going to cancel out of this equation on both sides. So it's a cubed times the coordinate volume times the energy density, which is row c squared. The rate of change of that is Du. And then on the right hand side, we have minus p minus b times dV, which is the rate of change of a cubed, again times the coordinate volume that we're talking about.

But that will cancel out on the two sides of the equation. So this is really just a description for the universe of the dU equals minus pdV equation. And this can just be rearranged, expanding the time derivatives, to give us row dot, and we get minus 3 a dot over a, times row plus p over c squared.

So this tells us how to relate row dot to the pressure. And it tells us that the formula that we started with a long time ago, which just said that row fell off like 1 over a cubed, was synonymous with saying the pressure is zero, for a gas of non-relativistic particles, the pressure is negligible. But for radiation, clearly if we're going to get a four instead of a three, the pressure will be non-negligible.

And in fact, it implies that the pressure is exactly equal to one third of the energy density for a gas of radiation. OK, knowing that, we can now look back at the Friedmann equations and ask, how do they stand up? Are they still consistent, or do we have to modify something? And this is really the crucial point.

What we know are these three equations, which are the two Friedmann equations and the equation for row dot. And we could see immediately that those equations are not independent of each other. The easiest thing to see is that if we start with the top equation, we could differentiate it. And since the top equation has a dot in it, when we differentiate, we'll get an equation for a double dot.

But the equation will also involve row dot, if we take the time derivative of that top equation. But if we know what row dot is, we could put that in, and in the end we'll get an equation for a double dot by itself. And it will in fact agree with the equation on the middle line. Again, things would be inconsistent. Things are consistent, we didn't make any mistakes.

If we derive the equation for a double dot by using the first and third of those equations, then we'll get the second of those equations. But the catch is that now we want to modify the third of those equations, the equations for row dot. And then the Friedmann equations as we've written them will not be consistent anymore. Because we'll have a different equation for row dot, we'll get a different equation for a double dot.

So we have to decide what gives. What can we change to make everything consistent? And here, the rigorous way of proceeding is to look at general relativity and see what it says, and the answer we're going to write down is exactly what general relativity says. But we can motivate the answer in, I think, a pretty sensible way, by noticing that as the universe expands, we'd expect the energy density to vary continuously, because energies are conserved.

And we also expect a dot to vary continuously, because basically, the mechanics of the universe are like ton's laws. And velocities don't change discontinuously. You can apply a force, and that causes velocities to have a rate of change. But velocities don't change instantaneously, unless you somehow apply an infinite force.

And the same thing will be true with the universe. On the other hand, accelerations can change instantaneously. You could change the force acting on a particle, in principle, as fast as you want, and the acceleration of the particle will change at that same rate. So if we look at these equations, we would expect that the first equation and the third equation would not be allowed to involve the pressure.

Because the pressure basically is a measure of a force. Pressures can change instantaneously. So what you need to do, if we're going to make these equations consistent in the presence of pressure-- which changes the row dot equation, the only equation we can change is the second one. And then we can ask ourselves, what do we have to change it to make the three equations consistent?

And this is what you looked at on your homework. And the answer is that the a double dot equation has to be modified to give the equation at the bottom of the screen here. And this is the correct form of the a double dot equation in cosmology. And this is what we'll be using for the rest of the term, this is exactly what you would get from general relativity.

As long as we're talking about homogeneous and isotropic universes, this formula as exact as far as we know. OK, any questions about that? Yes.

AUDIENCE: Why when we derive that equation do we use--

PROFESSOR: This equation?

AUDIENCE: Or the one above that.

PROFESSOR: Yeah.

AUDIENCE: We use dU equals minus pdV? I mean, I agree with that, but couldn't we use a more complete version? Like, the complete version of the first law of thermodynamics, that dU equals TDS minus pdV.

PROFESSOR: OK, yeah. The question was when we wrote down dU equals minus pdV, why did we not include a plus TDS term here, which could also be relevant. The answer is that for the applications we're interested in-- you're quite right, it could be important, but for the applications that we're interested in, which is the expanding gas in the universe, the expanding gas in the universe will be making use of this fact.

It really does expand adiabatically, that is, there's nothing putting heat in or out, and everything is remaining very close to thermal equilibrium, which means that entropy does not spontaneously change. So the TDS term we will be assuming is very, very small, and that's accurate. And you're right, if that were not the case, there would be further complications in terms of figuring out what row dot is.

Let me point out here that this equation actually does contain a somewhat startling perhaps fact about gravity it says that in the context of general relativity. And that's really the context that we're in, even though we haven't learned a lot of general relativity. But it says that in the context of general relativity, pressures, as well as mass densities, contribute to the gravitational field. A double dot is basically a measure of how fast gravity is slowing down the universe.

And this says that there's a pressure. It can also help to slow down the universe. Meaning that pressure itself can create a gravitational field. In the early universe, where we go back to this radiation dominated period, we know that the pressure is one third of the energy density. That says that this pressure term is the same size exactly as the mass density term.

So in the radiation dominated phase, the pressure is just as important in effect for slowing down the universe as is the mass density. In today's universe it's negligible. Well, we'll come back that. The dark energy has a non-trivial pressure, but the pressure of ordinary matter in today's universe is negligible.

The other important fact about this equation is that energy densities, so far as we know, are always positive. We don't know for sure what the ultimate laws of physics are, but for all the laws of physics that we know, energy densities are positive. On the other hand, pressures actually can be negative for some kinds of material.

And we'll talk a little bit more about how to get a negative pressure later. But this formula tells us that positive pressures act the same way as positive mass densities, creating an attractive gravitational field which slows down the expansion of the universe. But if there could be a material with a negative pressure, this same equation, which would presumably still hold, and believe it does, would tell us that that negative pressure would actually cause the universe to accelerate, because of its gravitational effects.

Now, we're not talking about the mechanical effects of the pressure. Mechanical effects of pressure only show up when there are pressure gradients, when the pressure is uneven. So the very large air pressure in this room, which really is quite large, we don't feel all, because it's acting equally in all directions. Uniform pressures do not produce forces. So the mechanical effects of the pressure in the early universe, which we're assuming is completely homogeneous, is zilch. There is no mechanical effect.

But what we're seeing in this equation is a gravitational effect caused by the pressure. And it's obviously gravitational, business the effect is proportional to Newton's capital G, a constant determining the strength of gravity. So the equation is telling us that a positive pressure creates a gravitational attraction, which would cause the universe to slow down in its expansion.

But a negative pressure would produce a gravitational repulsion, which would cause universe to speed up. And we now know that, today-- in fact, for the last five billion years or so-- our universe itself is actually accelerating under the influence of something. And the only explanation we have is that the something that's causing the universe to accelerate is the repulsive gravity caused by some kind of a negative pressure material.

And that negative pressure material is what we call the dark energy. And we'll talk a little more later about what it is. It's very likely just vacuum energy. But we'll come back to that later in the course. Yes?

AUDIENCE: When we looked at the toy example of the piston in the cavity, the pressure of the gas was pushing outwards against the wall of the container. But we can't have that view of the universe, really, because there's no exteriors in the universe.

PROFESSOR: There's no walls, right.

AUDIENCE: So how should we view pressure in the sense that--

PROFESSOR: OK. Yeah. OK, the question is, in our toy problem involving the piston, we had walls for the pressure to push against. And that was where the energy went. It went into pushing the walls. When we're talking about the universe, there are no walls. How does that analogy work? What plays the role of the walls?

And the answer, I think, is that the role of the walls, when we're talking about the universe, first of all, you can ignore it if you just took in a small region. You could still just say, the small region is pushing out on the regions around it. And I think that's enough to make the logic clear.

But it still leaves open the question of, ultimately, where does this energy go? So saying it goes from here to there doesn't help you unless you know where it goes after it goes from and there and there. So you might want to ask the question more generally, where does the energy ultimately end up?

And then I think the answer is that it ends up in gravitational potential energy. You could certainly build a toy model, where you just have a gas in a finite region, self-contained under gravity. And then you'd have to make up some kind of a mechanism to cause it to expand.

But when you cause it to expand, you'll be pulling particles apart, which are attracting each other gravitationally. And that means you'll be increasing the gravitational potential energy as you pull the gas apart. So I think, ultimately, the answer is the energy imbalance that we seem to be seeing here is taken up by the gravitational field so that, all in all, energy still conserved. Any other questions?

OK, in that case, let us continue on the blackboard. OK, first thing I want to look at it is just the behavior of a radiation-dominated flat universe. So a flat universe is going to obey H squared equals 8 pi over 3 G rho, and then the potential minus kc squared over a squared. Hard to write dotted lines on the blackboard. But this potential term is not there, because k equals 0.

We're talking about the flat case. So for a flat case, we could just express H in terms of rho. And we know how rho behaves for radiation. Rho falls off as 1/a to the fourth. So H squared is proportional to 1/a to the fourth. That means that H itself is proportional to 1/a squared.

So we can do that. a-dot over a is equal to some constant over a squared, a-dot over a being H. And now we can multiply both sides by a, of course. And we get a-dot is equal to a constant over a.

And this we can integrate. The way to integrate is to put all the a's on one side and all of the dt's on the other side. So we get ada, writing this as da/dt. So ada is equal to the constant times dt.

And then, as we've done before, when we're talking about matter, it's the same calculation there. I just did different power of a appears so we'd know how to do it. Integrating, we get 1/2a squared is equal to the constant times t, and then plus a new constant of integration, constant prime.

Now we make the same argument as we've made in the past. We have not yet said anything that determines how our clocks are going to be set. So we can choose to set our clocks in the standard way, which is to set our clocks so that t equals 0 corresponds to the moment where a is equal to 0.

And if a is going to be equal to 0 when t is equal to 0, it means constant prime is going to be equal to 0. So by choosing the value of constant prime, we really are just determining how we're going to set our clocks, how we're going to choose the 0 of time. And we'll do that by setting constant prime equal to 0.

And then we get the famous formula for a radiation-dominated universe, a of t is just proportional to the square root of t, or t to the 1/2 power. And this is for a radiation-dominated flat universe, replacing the t to the 2/3 that we have for the matter-dominated flat universe.

Once we know that a is proportional to the square root of t-- and for the flat universe, the constant proportionality mean nothing, by the way. It's not that we haven't been smart enough to figure out what it is. It really has no meaning whatever. You could set it equal to whatever you want, and it just determines your definition of the notch, your definition of how you're going to measure the comoving coordinate system.

Once we know this, we should know pretty much everything. So in particular, we can calculate h, which a-dot over a. And the constant proportionality drops when we compute a-dot over a. As we expect, it has no meaning. So it should not appear in the equation for anything that does have physical meaning.

So H is just 1/2t, the 1/2 here coming from differentiating the t the 1/2 power. We can also compute the horizon distance. So the physical horizon distance, l sub p horizon, where p stands for physical, is equal to the scale factor times the coordinate horizon distance. And the coordinate horizon distance is just the total coordinate distance that light could travel from the beginning of the universe.

And we know the coordinate velocity of light is c divided by a. So we just integrate that to get the total coordinate distance. So it's the integral from 0 to t of c over a of t prime dt prime. And since a of t is just t to the 1/2, this is a trivial integral to do. And the answer is 2 times c times t.

So in a radiation-dominated universe, the horizon's distance is twice c times t. For a static universe, the horizon distance would just be c times t. It would just be the distance light can travel in time t, but more complicated in an expanding universe. For the matter-dominated case, we discovered that the horizon distance was 3ct, if you remember. For the radiation-dominated case, it's 2ct.

And finally, an important equation is that, going back to here, where we started, this equation relates H to rho. We found out that, merely by knowing the universe is radiation-dominated, without even caring about what kind of radiation it is, how much neutrinos, how much photons, whatever-- doesn't matter-- merely by knowing the universe is radiation-dominated, we were able to tell that H is 1/2t. And if we know what H is, that formula tells us we also know what rho is.

So without even knowing what kind of radiation is contributing, we know that, for a radiation-dominated universe, rho is just equal to 3 over 32 pi Newton's constant G times time, little t, squared. It's rather amazing that we can write down that formula without even knowing what kind of radiation is contributing. But as long as that radiation falls off as 1 over the scale factor to the fourth, and as long as we know the universe is flat, then we know what that energy density has to be. This is crucial here, by the way. The energy could be anything if we did not assume that the universe was flat.

OK, any questions about this? Yes?

AUDIENCE: If we assumed that it was almost flat, would we be able to have any bounds on it?

PROFESSOR: OK, question is, if we assumed that it was almost flat, would we be able to have any bounds on it? The answer is, yeah, if you were quantitative about what you meant by "almost flat," you could know how almost true that formula would have to be.

OK, if there are no other questions, I want to switch gears slightly now and go back to talk about some of the basic underlying physics that we are going to need, and in particular, the physics of black-body radiation. So this is really just a little chapter of a stat mech course that we're inserting here, because we need it. And because it comes from another course, we're not going try to do it in complete detail.

But I'll try to write down formulas that make sense. And that will give us what we need to know to proceed. So that will be the goal.

So what is black-body radiation? The physical phenomenon is that, if one imagines a box with a cavity in it-- that's supposed to be a box with a cavity in it, in case you can't recognize the picture-- if the box is held at some uniform temperature t-- t is temperature-- then is claimed and verified experimentally that the cavity will fill up with radiation-- in this case, we're really just talking about electromagnetic radiation- - the calving will fill up with electromagnetic radiation whose characteristics would be determined solely by that temperature t and will therefore be totally independent of the material that makes up the box.

Roughly speaking, I think the way to think about it is to say that the box will fill up with radiation at temperature t. And saying that the radiation has temperature t is enough to completely describe the radiation. It doesn't matter what kind of a box that radiation is sitting in. So the box will fill with radiation at temperature t. And that radiation is called black-body radiation.

Like many things in physics, it has a variety of names, just to confuse us all. So it's also called cavity radiation, which makes a lot of sense, given the description we just gave. And it's also sometimes called just thermal equilibrium radiation. This is radiation at temperature t.

I haven't really justified the word "black-body" radiation yet, so let me try to do that quickly. The reason why it can be called black-body radiation-- and this will be important for some things in cosmology; I'm not sure if it will be important to us or not, but certainly important to know-- the reason why it's called black-body radiation is because we imagine inserting into this cavity a black body, in the literal sense. What is the literal sense of a black body? It's a body which is black in the sense that all radiation that hits it is absorbed.

Now, this black body is still going to glow. If you heat a piece of iron or something to very high temperatures, you see it glow. That glow is not reflection. That glow is emission by the hot atoms in the piece of iron or whatever.

Emission is different from reflection. When we say it absorbs everything, we mean it does not reflect anything. But it will still admit by thermal de-excitation.

The crucial distinction between reflection and thermal emission is that reflection is instantaneous. When a light beam comes in, if it's reflected, it just goes back out instantaneously. Emission, thermal emission, is a slower process. Atoms get excited, and eventually, they de-excite and emit radiation. So it takes time. And that's the distinction. We're going to assume that this body is black in the sense that there's no reflection.

OK, now we're going to make use of the fact that we know that thermal equilibrium works. That is, if you let any isolated system sit long enough, it will approach a unique state of thermal equilibrium determined by its constituents, which you've put in to begin with, but otherwise independent of how exactly you arrange those constituents. So if we put in, for example, a cold black body, it will start to get harder, warming up to the same temperature as everything else. If we put in an extra hot black body, it would emit energy and start to cool down to the temperature of everything else. But eventually, this black body will be at the same temperature as everything else.

And we're going to be assuming here that the box itself is being held at some fixed temperature t. So wherever energy exchange occurs because of this black body, it will be absorbed by whatever is holding the outer box at the fixed temperature. So in the end, if we wait long enough, this black body is going to acquire the same temperature as everything else and hold that temperature. Now, if it's holding that temperature, it means that the energy input to the box, to the black body, will have to be the same as the energy output of the black body.

Now, the black body is going to be absorbing radiation, because we have radiation here, and we said that any radiation that hits it is absorbed. That was the definition of "black." So it's clearly absorbing energy.

If it's not going to be heating up-- and we know that it's not, because it's in thermal equilibrium; the temperature will remain fixed-- in order for it to not heat up, it has to radiate energy, as well. And the energy it radiates has to be exactly the same as the energy it's absorbing once it reaches thermal equilibrium. So in equilibrium, the black body, BB, radiates at same rate that it absorbs energy.

This radiation process is this slow process of thermal emission. There are atoms inside this black body that are excited. Those atoms will de-excite over time, releasing photons that will go off.

And the important thing about that slow mechanism is that, if we imagine taking this black body out of its cavity, but not waiting long enough for its temperature to change-- so we'll assume its temperature is still the same, t. So this is a picture of the same black body at temperature t, but now outside the cavity. Its radiation rate is not going to change when we take it outside the cavity, because the radiation was caused by things happening inside the black body, which are not changed when we put it in or out of the cavity.

So it will continue to radiate at exactly the same rate that it was radiating when it was in the cavity. And that means it's going to radiate at exactly the same rate as the energy that it would have absorbed if it were bathed by this black-body radiation. So essentially, it means it will emit black-body radiation with exactly the intensity that the black-body radiation would have on the outside if the black body were still inside the cavity. So it radiates with exactly the same intensity as the energy that it would receive if it were inside the cavity.

And furthermore, you could even elaborate a bit on his argument to show that the radiation that it radiates has exactly the same spectrum, exactly the same decomposition into wavelengths, as the black-body radiation inside the cavity. And the way to see that is to imagine surrounding this black body by absorption filters that only let through certain frequencies. And the point is that, no matter what frequencies you limit going through this filter, you have to stay in equilibrium. It will never get hotter or colder.

So that means that each frequency by itself has to balance, has to have exactly the same emission as it would have abortion if the black body were just exposed to black-body radiation surrounding it. So it radiates black-body radiation. And the intensity and spectrum must match what we call black-body or cavity radiation. So the cavity radiation has to exactly mimic the radiation emitted by this black body. And that's the motivation for calling it black-body radiation.

Now, if this black body absorbed some radiation and reflected some, then it would emit different radiation. So it is important that this body be black, in the sense that it doesn't reflect anything. All radiation hitting it is absorbed. And only under that assumption do we know exactly what it's going to emit. Yes?

AUDIENCE: So is this only true for right after you take it out?

PROFESSOR: Well, it will start to cool after you take it out. And as it cools, its temperature will change. But if you account for the changing temperature, it will be true at anytime, actually. But the temperature will change.

AUDIENCE: Because, in the black cavity, it has things exciting it. And when you take it out, there's no photons, no constant radiation to excite it. So it can radiate--

PROFESSOR: Yes. Once you take it out, it's no longer being excited. And I think, technically, you're right. Once you take it out, it will not only cool, but it will cease to be at a uniform temperature. And that's basically what we're saying if we're saying that the atoms that are excited won't necessarily be in the right thermal distribution as they would be if it was on the inside. That would be a statement that is not any longer in thermal equilibrium.

But as long as the radiation is slow, you could just account for the changing temperature. You would know how it radiates. And I think that's a very good approximation. Although in principle, it will cease to be in thermal equilibrium, as soon as you take it out, the edges will be cool, and the center will be hot. And you'd have to take into account all of those things to be able to understand how it radiates.

Any other questions?

OK, next, I want to talk a little bit about what this black-body radiation is. And one can begin by trying to understand it purely classically, which, of course, is what happened historically. In the 1800s, people tried to understand cavity radiation or black-body radiation using classical physics, Maxwell's equations, to describe the radiation.

And then, in a nutshell-- we're just trying to establish basic ideas here-- one can try to treat a field statistical-mechanically by imagining not fields in empty space, but fields in some kind of a box. In this case, it doesn't necessarily have to be the cavity that we're talking about. It could be a big box that just enclosed the system somehow to make it easier to talk about. And in the end, you could imagine taking that box to infinity, this theoretical box that you use to simplify the problem.

But once you put the system in a box, then a field, like the electromagnetic field, can always be broken up into normal modes, standing wave patterns that have an integer or a half integer number of wavelengths inside the box. And no matter how complicated the field is inside the box, you could always describe it as a superposition of some set of standing waves. In general, it takes an infinite number of standing wave components to describe an arbitrary field-- that is, with shorter and shorter wavelengths. But you can always-- and this is Fourier's theorem-- you can always describe an arbitrary field in terms of the standing waves.

And that's good for the point of view of thinking about statistical mechanics, because you could think about each standing wave almost as if it were a particle. It really is a harmonic oscillator. So if you think you know the statistical mechanics of harmonic oscillators, each standing wave in the box is just a harmonic oscillator, so simple.

We now try to ask what is the thermodynamics of this system of harmonic oscillators. And the rule for harmonic oscillator is simple. Stat mech tells you that you have 1/2 kT per degree of freedom in thermal equilibrium. The energy of a system should just be 1/2 kT per degree of freedom.

Having said that, all the complicate questions come about by asking ourselves what is meant by degree of freedom. But for the harmonic oscillator, that has a simple answer. A harmonic oscillator has two degrees of freedom-- the kinetic energy and the potential energy. So the energy of a harmonic oscillator should just be kT per degree of freedom.

And we could apply that to our gas in the box and we could, ask how much energy should the gas absorb at a given temperature? What should be the energy density of the gas at a given temperature-- this gas of photons.

But it was noticed in the 1900-- the 1800s that this doesn't work because there's no limit to how short the wavelengths can be. And therefore, there's not just some finite set of harmonic oscillators. There's an infinite set of harmonic oscillators where you have more and more harmonic oscillators at shorter and shorter wavelengths ad infinitum, no limit. And that came to be known as Jean's Paradox.

So what it suggests is that if this classical stat mech worked-- which obviously it's not working. But if it did work, it would mean that as you tried to put a gas in just an empty box in contact with something at a fixed temperature, the box would absorb more and more energy without limit. And ultimately, it would presumably cause the temperature of the whatever is trying to maintain the temperature to go to 0 as energy gets siphoned off to shorter and shorter wavelengths of excitations.

That obviously isn't the way the world behaves. We'd all freeze to death if it did. So something has to happen to save it. And it wasn't at all obvious for many years what it was that saves it. But this Jean's Paradox turns out to be saved by quantum mechanics.

And the important implication of quantum mechanics is that the energy of a harmonic oscillator is no longer allowed to have any possible value, but is now quantized as some integer times h times nu, the frequency-- h being Planck's constant, nu being frequency, n being some integer where this integer might be called the excitation level of the harmonic oscillator.

Depending how you choose your 0, you might have an n plus 1/2 there. But that's not important for us right now. It will be important later actually. But for now, we'll just allow ourselves to readjust the 0 and just think of it as n times h nu, or h bar omega.

Now, this makes all the difference, statistical mechanically. One can apply statistical mechanics using basically the same principles to the quantum mechanical system. And the key thing now is that for the very short wavelengths, which are the ones that were giving us trouble-- the infinities came to short wavelengths. For the short wavelengths where nu is high, h nu is high.

And it means that there's a minimum ante that you could put in to excite those short wavelength harmonic oscillators. And it's a large number. You either put in a large amount of energy or none at all. Quantum mechanics doesn't let you do anything in between.

Now remember, the classical mechanics answer was that you have kT in each harmonic oscillator. And kT would be small compared to h nu, if we're talking about a very short wavelength. So the classic answer is just not allowed by quantum mechanics. You either have to put in nothing or an amount much, much larger than the classical answer.

And when you do the statistical mechanics quantum mechanically, which is not a big deal really, you find that when you're confronted with that choice, the most likely answer is to put in no energy at all. So quantum mechanics freezes out these short wavelength modes. And then the n produces a finite energy density for a gas of photons. Yes?

AUDIENCE: Seems like if you were to sum over like all the possible wave numbers, that-- well, so the energy is inversely related to wavelength, right? So even if you quantize it, like for large wavelengths, isn't the sum still like a sum of one over lambda wavelength, with its derivative?

PROFESSOR: You're saying, isn't there also a divergence at the large wavelength n?

AUDIENCE: Because that sum doesn't seem like it would work.

PROFESSOR: Right. No, that's important. The reason that's not a problem is that if you're talking about the energy in a box, the wavelength can't be bigger than the box. The largest possible wavelength is twice the box so that half a wavelength fits in the box. If you're talking about the energy in the whole infinite universe, then we expect the answer to infinite. And it is. There's no problem with having infinite total energy if you want to have a finite energy density throughout an infinite universe.

So the size of the box cuts off the large wavelengths. And quantum mechanics cuts off the small wavelengths. So in the end, one does get a finite answer for the energy density of black-body radiation. And that's crucial for our survival, crucial for the existence of the universe as we know it, and also crucial for the calculations that we're about to do.

OK, so when one does these calculations initially for photons only, what we'd find is that the energy density is equal to a fudge factor, which I'm going to call g. And you'll see later why I'm introducing a fudge factor. For now, g is just 2. But later, we'll generalize the application of this formula, and g will have different values. But for now, we're dealing with photons. There's a factor of 2 there, but I'm going to write 2 as g, writing g equals 2 underneath.

And then the pi squared over 30-- you can really calculate this-- times kT to the fourth power divided by h bar c cubed, h bar being Plank's constant divided by 2 pi and little k being Boltzmann constant. So this is calculated just by thinking of the gas in a box as a lot of harmonic oscillators and applying standard stat mech to each harmonic oscillator, but you apply the quantum mechanical version of the stat mech to each harmonic oscillator.

And you can also find, by doing the same kind of analysis, that the pressure is 1/3 the energy density, which we also derived earlier by different means. And it's all consistent so you get the same answer every time, even if you think about it differently. So here, I have mine deriving it directly from the stat mech.

You can also, from the stat mech, calculate the number density of photons in thermal equilibrium. And that will be equal to -- again, there's a factor of 2. But this time, I'll call the factor of 2 g star, where g star also equals 2 for photons. But when we generalize these formulas, g will not necessarily equal g star, which is why I'm giving it two names.

And this g star multiplies zeta of 3, where zeta refers to the Riemann zeta function, which I'll define in a second, divided by pi squared times kT cubed divided by h bar c cubed. OK. OK, so I need to define this zeta of 3. It's 1 over 1 cubed plus 1 over 2 cubed plus 1 over 3 cubed plus dot dot dot. It's an infinite series. And if you sum up that infinite series, at least to three decimal places, it's 1.202.

OK, then there's one other formula that will be of interest to us. And that'll be a formula for the entropy density. Now, if you've had a stat mech course, you have some idea of what entropy density means. If you have not, suffice it to say for this class that it is some measure of the disorder in the sense of the total number of different quantum states that contribute to a given macroscopic description. The more different microstates there are that contribute to a macroscopic description, the higher the entropy.

And the other important thing about entropy to us besides that vague definition-- which will be enough-- but the important thing for us is that under most circumstances, entropy will be conserved. It's conserved as long as things stay at or near thermal equilibrium. And in the early universe as the universe expands, that's the case. So for us, the entropy of our gas will simply be a conserved quantity that we can make use of. And we will make use of it in some important ways.

And we could write down a formula for the entropy density of photons. And it's g, where this g in fact the same g as over there. It is related to the energy. So it's the same g that appears in two cases, 2 in both cases for protons by themselves. And then there are factors that you can calculate-- 2 pi squared over 45 times k to the fourth T cubed over h bar c cubed.

OK, this time, the number of k's and T's do not match. That's mainly due to the conventions about how entropy is defined. It's not really anything deep.

I might mention at this point that the 2's that I've been writing for everything-- g equals 2, g star equals 2-- the reason those 2's are written explicitly rather than just absorbing the factor of 2 into the other factors is that photons are characterized by the fact that there are two polarizations of photons. So if I have a beam of photons, they could be right-handed or left-handed. And anything else could be considered a superposition of those two.

So there are two independent polarizations. And it's useful to keep track of these formulas as the amount of energy density per polarization. Thus, different kinds of particles will have different numbers of polarizations. So if we know the amount per polarization, we'll be able to more easily apply it to other particles. Yes?

AUDIENCE: Sorry, just to kind of bring up the same question, if we-- I read that in the early universe, the temperature is constantly changing or it's cooling.

PROFESSOR: Right.

AUDIENCE: So then if the temperature is changing, then how can we say that the entropy is constant?

PROFESSOR: OK, important question-- we'll be getting to it very soon. But since you asked the question, I'll ask it now. The question was if the universe is expanding, and the entropy density is going down because it thins, how can that happen-- I guess it was asked the other way around. If the temperature is falling, how can entropy be conserved if this is the formula for entropy density? And the answer-- when I tell you, you'll see it's obvious.

We don't expect the entropy density to be conserved if the entropy is conserved. The entropy thins out as the universe expands. So if we just had a gas with nothing else changing, we expect the entropy density to go down like 1 over the scale factor cubed, just like the number density of particles.

So if s is going to go down like 1 over the scale factor cubed, that would be consistent with this formula if the temperature also fell as 1 over the scale factor. So that to cubing it made things match. And that's what we'll find. The temperature falls off, like 1 over the scale factor. And that's consistent with everything that we said about energy densities and so on.

OK. Next thing I want to talk about is neutrinos, which I told you earlier contributes in a significant way to the radiation energy density in the universe today. Neutrinos are particles which for a long time, were thought to be massless. Until around 2000 or so, neutrinos were thought to be massless.

Now we know that in fact, they have a very small mass, which complicates the description here. It turns out that cosmologically, neutrinos still act as if they were massless for almost all purposes and for really all purposes that we'll be dealing with in this class, although if we were interested in the effects of neutrinos on structure formation, we'd be interested in whether or not the neutrinos have a small mass or whether it's smaller than that. We know it's non-zero.

We don't know what the neutrino mass is, by the way. What we actually know from observations is that there are three types of neutrinos. And those are called flavors. And I'll use the letter nu for the word neutrino. And those three types of neutrinos are called nu sub e, called the electron neutrino; nu subbed mu, called the muon neutrino, and nu sub tau, called the tau neutrino.

And these letters e, mu, and tau link to the names of particles. This is the electron neutrino. This is the muon neutrino connected to a particle called the muon, which is like the electron but heavier and different. And this is linked to a particle called the tau, which is also like the electron but much more heavier but otherwise similar in its properties.

And the neutrinos are linked in the sense that when a neutrino is produced, depending on how you start, it is very typically produced in conjunction with one of these other particles. So an electron neutrino is typically produced in conjunction with an electron. And similarly, a muon neutrino is typically produced in conjunction with a muon. And a tau neutrino is typically produced in conjunction with a tau.

Now, what does this have to do with neutrino masses? We've never actually measured the mass of neutrino. So we only know that they have mass indirectly.

What we have seen is one flavor of neutrino turn into another flavor. And it turns out, in the context of quantum field theory and I think this does make a certain amount of sense just by intuition, if a particle is massless, it can never change into anything. The process by which one changes into another is pretty quantum mechanical and a little hard to understand anyway.

But if the particles were really massless, they would move at the speed of light. And if the particles were moving at the speed of light, if the particle had any kind of a clock on the particle, that clock would literally stop with the particle moving at the speed of light. So if particles are massless, any internal workings that that particle might have to be frozen. That is, if it's a clock, it has to be a clock that's stopped completely.

And for reasons that are essentially that, although they can be made more formal and more rigorous, a truly massless particle could never undergo any kind of change whatever. It would have to stay exactly like it looks like to start with. Because it just has no time.

So the fact that these neutrinos turn into each other implies that they must have a nonzero mass. It must not really be moving at quite the speed of light. And that's the way the formalism works. And we could set limits on the masses based on what we know about the transitions between one kind of neutrino and another. Yes?

AUDIENCE: How do we explain photon decaying to an electron-positron pair then?

PROFESSOR: A photon decaying to an electron-positron pair?

AUDIENCE: Yeah.

PROFESSOR: The answer is a free photon never does decay to an electron-positron pair. Photons can collide with something and produce an electron-positron pair. But that collision, that's a more complicated process.

What I'm saying is-- I'm sorry. This process of conversion-- maybe I should have clarified-- happens just as the neutrinos travel. It's not due to collisions. Due to collsions, complicated things can happen whether the particle is massless or not.

But a massless particle simply in transit cannot undergo any kind of transition. And these neutrinos are seen to undergo transitions simply being in transit without any collisions. In terms of my clock analogy and a stopped clock, I think the reason the photon can convert into electron-positron pairs if it collides with something is that when it collides, it essentially breaks the clock. You don't have a photon that's just moving along without time anymore.

OK, so these neutrinos have masses. And maybe I should, at this point, write down some bounds on these masses. m squared 21 times c to the fourth is equal to 7.50 plus or minus 0.2 times 10 to the minus 5 electron volts squared. These numbers come from the latest particle data tables, which I gave references for in the notes.

So this is the difference of the mass squared. And delta m 23 squared times c to the fourth to turn it into the square of an energy is 2.32 plus 0.12 minus 0.08 times 10 to the minus third electron volts squared. So one thing you notice immediately is that these are incredibly small masses. Remember, the proton weighs 938 MeV, three million electron volts. And these are fractions of one electron volt.

So by the standards of particle physics, these are unbelievably small energies, unbelievably small mass differences. But they're there. They have to be there for the physics we know to make sense.

The other thing that you may notice about this notation-- which I don't want to elaborate on but I'll just mention-- this is called 21. This is called 23. There's no 1, 2, or 3 there. There's an e and a mu and a tau. The complication here is something very quantum mechanical. The e, mu and tau labels are labels which basically label the neutrino according to how the neutrino is created.

It turns out that their mass eigenstates-- states which actually have a definite mass-- are not the e, the mu or the tau. In fact, if the e had a definite mass, that would be saying that an e would just propagate as a particle with a certain mass. It would not convert into anything else. The fact that an e converts into other particles-- a nu sub e converts into other particles is really the statement that nu sub e is not a state with a definite mass.

But there are states with definite masses which could be expressed quantum mechanically as superpositions of these flavor eigenstates. So nu sub 1, nu sub 2 and nu sub 3 are states of neutrinos that have definite masses. And each one of them is a superposition of nu sub e, nu sub mu and nu sub tau. Yes?

AUDIENCE: How come we don't have a delta n from 31?

PROFESSOR: Good question. I think it's just the lack of knowledge. I don't think there's any reason it's not defined. I'm sure it is defined. I think it's just lack of knowledge. And if I knew more details about how these things were measured, I could give you a better story about that. But I don't, frankly.

So in the end, it's a rather complicated quantum mechanical system which we're not going to go into any details about. What more should I say today?

OK, let me just mention for today and we'll continue next time after the quiz, for our purposes, we're going to treat these neutrinos as if they're massless. And it turns out that that's actually extraordinarily accurate from the point of view of cosmology, at least for the kind of cosmology that we're doing where we're just interested in the effect of these neutrinos on the expansion rate of the universe.

And treating them as massless particles, I will shortly give you the formulas for how they contribute to the black-body radiation. But it I think there's no point in my writing them now. I'll just write them again in the beginning of next period.

But they do contribute to the black-body radiation and in a way that we actually know how to calculate. And they have a noticeable effect on the evolution of our universe. OK, that's all for today. Good luck on the quiz on Thursday. I'll be here to help proctor. I think Tim [INAUDIBLE] will be here too. And I'll see you more intimately either at my office hour tomorrow or at lecture a week from now.