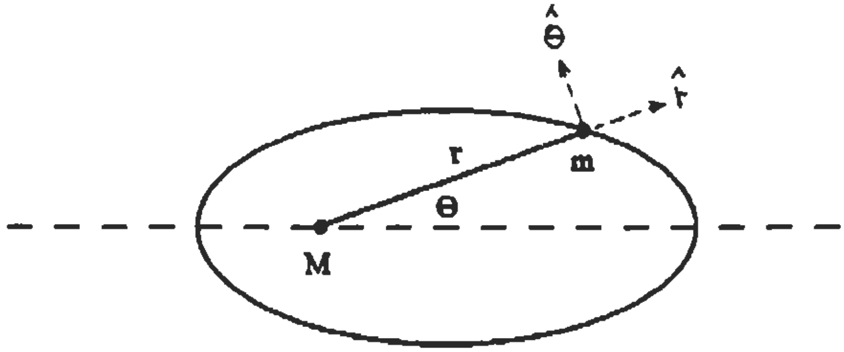
Kepler Problem - 8.282

Kepler problem

A mass m orbits a much larger mass M in a non-circular orbit. We want to find $r(t)$ and $\theta(t)$.



Start with $\vec{F} = m\vec{a}$ in the two dimensions of the orbital plane.

$$F_r = -\frac{GmM}{r^2} = -m\left(\frac{d\theta}{dt}\right)^2 r + m\frac{d^2r}{dt^2}$$

$$F_\theta = 0 \Rightarrow \text{Conservation of orbital angular momentum, } L$$

but, $\quad L = m v_\perp r = m\omega r r = m\omega r^2 = m\left(\frac{d\theta}{dt}\right)r^2$

Substitution #1: Eliminate the $\left(\frac{d\theta}{dt}\right)$ term from the equation for F_r which, in its present form, involves r, θ, and t

$$-\frac{GmM}{r^2} = -m\left(\frac{L}{mr^2}\right)^2 r + m\frac{d^2r}{dt^2}$$

or $$\boxed{\frac{d^2r}{dt^2} + \frac{GM}{r^2} - \frac{L^2}{m^2r^3} = 0}$$

This equation turns out to be difficult to solve. We will therefore use the chain rule of differentiation to find an equation for $r(\theta)$ instead of $r(t)$. This will yield the shape of the Keplerian orbit.

From our conservation of angular momentum expression

$L = m\left(\frac{d\theta}{dt}\right) r^2$ we have $\frac{d\theta}{dt} = \frac{L}{mr^2}$

Thus, $\frac{dr}{dt} = \left(\frac{dr}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = \frac{L}{mr^2}\left(\frac{dr}{d\theta}\right)$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left[\frac{L}{mr^2}\left(\frac{dr}{d\theta}\right)\right] = \left(\frac{d\theta}{dt}\right)\frac{d}{d\theta}\left[\frac{L}{mr^2}\left(\frac{dr}{d\theta}\right)\right]$$

Finally, $\frac{d^2r}{dt^2} = \frac{L^2}{m^2}\frac{1}{r^2}\frac{d}{d\theta}\left[\frac{1}{r^2}\frac{dr}{d\theta}\right]$

The equation for $r(\theta)$ can now be written as follows

With Substitution: #2

$$\frac{L^2}{m^2}\frac{1}{r^2}\frac{d}{d\theta}\left[\frac{1}{r^2}\frac{dr}{d\theta}\right] + \frac{GM}{r^2} - \frac{L^2}{m^2r^3} = 0$$

or $$\boxed{\frac{d}{d\theta}\left[\frac{1}{r^2}\frac{dr}{d\theta}\right] + \frac{GMm^2}{L^2} - \frac{1}{r} = 0}$$

It is not obvious that this equation for $r(\theta)$ is easier to solve than our original equation for $r(t)$, but it is!

The trick is to make the substitution of variables $r \equiv \frac{1}{u}$

With Substitution: #3

$$\frac{d}{d\theta}\left[u^2\frac{d}{d\theta}\left(\frac{1}{u}\right)\right] + \frac{GMm^2}{L^2} - u = 0$$

or $$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2}}$$

First, set the right hand side equal to zero to yield an equation

$$\frac{d^2u}{d\theta^2} + u = 0$$

This equation has solutions of the form:

$\underline{u(\theta) = B\cos(\theta+\beta)}$, where B and β are constants that are set by the boundary conditions

(Verify this solution for yourself)

To find the solution with the right hand side not equal to zero, simply add $\frac{GMm^2}{L^2}$ to the solution found above

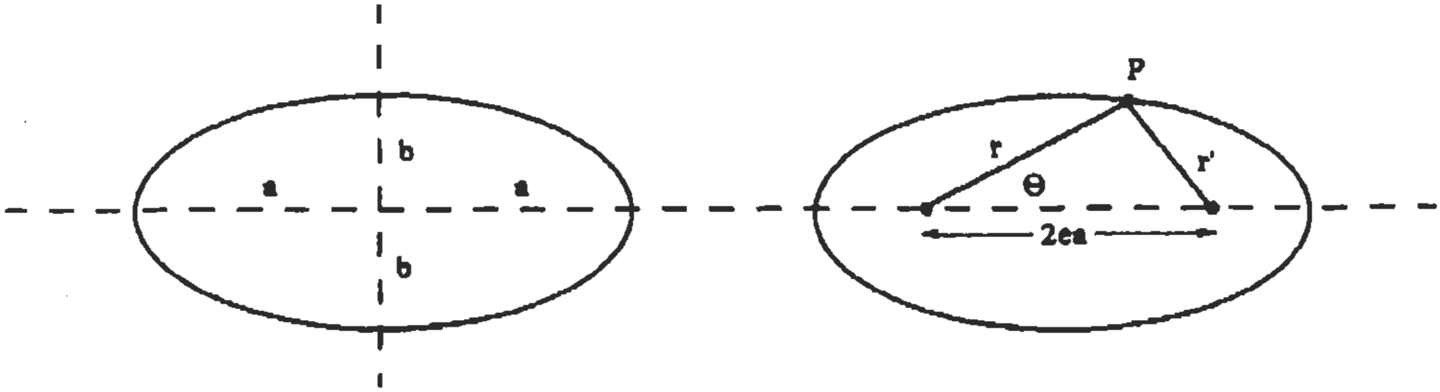
Thus $$u(\theta) \equiv \frac{1}{r(\theta)} = B\cos(\theta+\beta) + \frac{GMm^2}{L^2}$$

If we orient our plot of $r(\theta)$ so that the minimum and maximum values of r occur along the X axis, this fixes $\beta \equiv 0$.

Finally, $$\boxed{r(\theta) = \frac{L^2/GMm^2}{1 + \frac{BL^2}{GMm^2}\cos\theta}}$$

The constant B has yet to be determined.

Our expression for $r(\theta)$ is that of an ellipse. Therefore, let us explore some of the properties of ellipses.



The drawing on the left shows an ellipse with its semimajor and semiminor axes of lengths a and b, respectively. The drawing on the right shows the familiar geometrical construction of the same ellipse, where a string of fixed length is pulled taut around two fixed pins (defining the two foci) and a pencil at point P. The separation of the two foci are, by definition, equal to 2ea, where e is the orbital eccentricity.

We can now derive the equation of an ellipse in r and θ coordinates.

It is easy to show that the length of the string is $2a + 2ea = 2a(1+e)$

We now employ the law of cosines on the drawing on the right:

$$r'^2 = r^2 + (2ea)^2 - 4ear\cos\theta$$

and string length $= r + r' + 2ea = 2a + 2ea \Rightarrow r + r' = 2a$

$$(2a-r)^2 = r^2 + (2ea)^2 - 2ear\cos\theta$$

$$4a^2 - 4ar + \cancel{r^2} = \cancel{r^2} + 4e^2a^2 - 4ear\cos\theta$$

$$a^2(1-e^2) = ar(1-e\cos\theta)$$

Finally, we find the equation of an ellipse in polar coordinates:

$$\boxed{r(\theta) = \frac{a(1-e^2)}{(1-e\cos\theta)}}$$

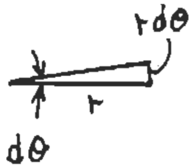
This is exactly the same form that we derived from the Keplerian equation of motion! By equating the numerators of these two expressions we find:

$$a(1-e^2) = L^2/GMm^2$$

Thus, we see that for a given set of masses, if we choose the semimajor axis, <u>a</u>, and the orbital angular momentum, L, this sets the orbital eccentricity. This concludes the proof of <u>Kepler's first law</u>.

<u>Kepler's second and third laws</u> easily follow:

Recall that $L = m\left(\frac{d\theta}{dt}\right) r^2$ = constant



but, $\frac{1}{2}(r\,d\theta)\,r$ = the differential area swept out by the orbit

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{L}{2m}$$

Consequence of conservation of angular momentum only, not the $1/r^2$ law of gravity.

Finally, we have $\boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{constant}}$ Kepler's 2^{ND} law

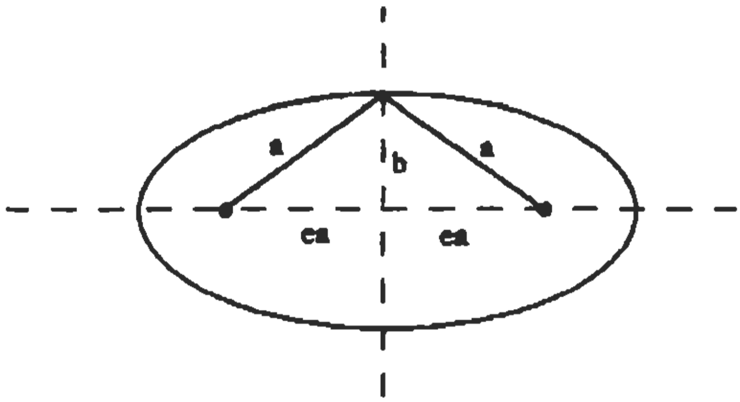
If we integrate the expression for swept-out area around the orbit, we should arrive at the area of the ellipse

$$\int_0^P \left(\frac{dA}{dt}\right) dt = \text{Area of ellipse} = \frac{L}{2m} P = \underbrace{\pi a b}_{\text{area of ellipse in terms of semimajor and semiminor axes}}$$

From the drawing, one can see that a, b, and e are related by

$$\frac{b^2}{a^2} = 1 - e^2$$

Therefore, we can arrive at the following relation between orbital period, P, and semimajor axis a :



$$\frac{L^2 P^2}{2^2 m^2} = \pi^2 a^2 b^2 = \pi^2 a^4 (1-e^2) = \pi^2 \frac{a^3 L^2}{GMm^2}$$, where we

have utilized the expression $a(1-e^2) = L^2/GMm^2$ from page 5.

Finally, we are left with

$$\frac{P^2}{(2\pi)^2} = \frac{a^3}{GM} \quad \text{or} \quad \boxed{\frac{GM}{a^3} = \left(\frac{2\pi}{P}\right)^2}$$ Kepler's third law for eccentric orbits

This is exactly the same form as for a circular orbit, except here a is the semimajor axis.

Time Dependence of the Orbital Motion

Return to the original second order differential equation in $r(t)$ that we started with:

$$\boxed{\frac{d^2r}{dt^2} + \frac{GM}{r^2} - \frac{L^2}{m^2r^3} = 0} \qquad \text{Force Equation}$$

but, $\frac{d^2r}{dt^2}$ can be written as $\frac{1}{2}\frac{d}{dr}(v_r^2)$, where v_r is the radial velocity.

With this, we can integrate the above equation (with forces and accelerations) to yield an energy equation:

$$\frac{1}{2}v_r^2 - \frac{GM}{r} + \frac{L^2}{2m^2r^2} = \text{const}$$

At periastron, $v_r = 0$, $r = a(1-e)$, and $L^2 = GMm^2a(1-e^2)$ (everywhere).

Thus, $\text{const} = -\frac{GM}{2a}$

$$\boxed{\frac{1}{2}v_r^2 - \frac{GM}{r} + \frac{L^2}{2m^2r^2} = -\frac{GM}{2a}} \qquad \text{Energy Conservation}$$

or $$v_r = \frac{dr}{dt} = \sqrt{-\frac{GM}{a} + \frac{2GM}{r} - \frac{L^2}{m^2r^2}}$$

or $$dt = \frac{r\,dr}{\sqrt{-\frac{GM}{a}r^2 + 2GMr - L^2/m^2}}$$

Now substitute $L^2 = GMma(1-e^2)$

$$dt = \frac{r\,dr}{\sqrt{\frac{GM}{a}}\sqrt{-r^2+2ra-a^2(1-e^2)}}$$

$$\boxed{dt = \frac{r\,dr}{\sqrt{\frac{GM}{a}}}\;\frac{1}{\sqrt{a^2e^2-(r-a)^2}}}$$

The equation in this form can be integrated by making the following (quite standard) trig substitution:

Let $\quad (r-a) = -ae\cos u$

$\qquad dr = ae\sin u\,du$

$$dt = \frac{a(1-e\cos u)(ae\sin u)\,du}{\sqrt{\frac{GM}{a}}\sqrt{a^2e^2-a^2e^2\cos^2 u)}}$$

$$dt = \frac{a^2e\,(1-e\cos u)\cancel{\sin u}\,du}{\sqrt{\frac{GM}{a}}\,ae\,\cancel{\sin u}}$$

Recall:

$$\left(\frac{2\pi}{P}\right)^2 = \frac{GM}{a^3}$$

$$\frac{2\pi\,dt}{P} = (1-e\cos u)\,du$$, which is trivial to integrate.

$$\boxed{\phi \equiv \frac{2\pi(t-T_0)}{P} = u - e\sin u}$$

mean anomaly

eccentric anomaly

And recall: $\boxed{r = a(1-e\cos u)}$ (from our trig substitution above)

The above two equations represent parametric solutions for $r(t)$, namely $r(u)$ and $u(t)$.

When fitting Keplerian orbits to observational data, one usually writes the parametric equations as $x(u)$ and $y(u)$ with the same $u(t)$ equation.

We can use our elliptical orbit solutions

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

to find $x(u)$ and $y(u)$. (Note that there is a + sign in the denominator, which is opposite to the sign derived earlier. This simply flips the ellipse 180° in orientation.) From the equation for the ellipse, we find

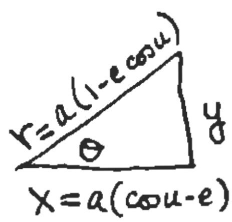
$$r + re\cos\theta = a(1-e^2)$$

or $\quad r\cos\theta \equiv x = [a(1-e^2) - r]/e$

substituting $\quad r = a(1-e\cos u)$ from above, we find

$$x(u) = [a(1-e^2) - a(1-e\cos u)]e = ae(\cos u - e)/e$$

$$\boxed{x(u) = a(\cos u - e)}$$



Similarly $\quad \boxed{y(u) = a\sqrt{1-e^2}\,\sin u}$

and we had $\quad \boxed{\frac{2\pi(t-T)}{P} = u - e\sin u}$

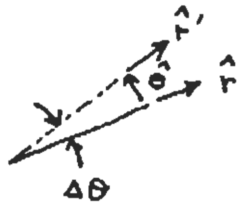
These non-linear equations are used to fit orbits where $x(t)$ and $y(t)$ have been measured. Basically, for any observation time t, one solves the bottom expression by Newton's method for u. Then $x[u(t)]$ and $y[u(t)]$ are evaluated from the top two equations and compared with the measurements.

Acceleration in a Plane

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

So, let $\vec{r} = r\hat{r}$, where $\hat{r}$ is the unit vector in the radial direction

Therefore, the velocity $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d}{dt}\hat{r}$, by the chain rule



$$\frac{\Delta\vec{r}}{r} = \Delta\theta\,\hat{\theta}$$

or $$\frac{\Delta\hat{r}}{\Delta t} = \frac{\Delta\theta}{\Delta t}\hat{\theta} \Rightarrow \frac{d\hat{r}}{dt} = \omega\,\hat{\theta}$$

So, $$\vec{v} = \frac{dr}{dt}\hat{r} + \omega r\,\hat{\theta}$$

Finally, $$\vec{a} = \frac{d\vec{v}}{dt}$$

or $$\vec{a} = \frac{d^2r}{dt^2}\hat{r} + \left(\frac{dr}{dt}\right)\left(\frac{d\hat{r}}{dt}\right) + r\left(\frac{d\omega}{dt}\right)\hat{\theta} + \left(\frac{dr}{dt}\right)\omega\,\hat{\theta} + \omega r\frac{d}{dt}\hat{\theta}$$

From a derivation similar to the one above for $\frac{d\hat{r}}{dt}$, we find

$$\frac{d\hat{\theta}}{dt} = -\omega\,\hat{r}$$

$$\vec{a} = \frac{d^2r}{dt^2}\hat{r} + \omega\left(\frac{dr}{dt}\right)\hat{\theta} + r\left(\frac{d\omega}{dt}\right)\hat{\theta} + \left(\frac{dr}{dt}\right)\omega\,\hat{\theta} - \omega^2 r\,\hat{r}$$

Collecting terms, we have

$$\boxed{\vec{a} = \left[\left(\frac{d^2r}{dt^2}\right) - \underbrace{\omega^2 r}_{\text{centripetal acceleration}}\right]\hat{r} + \left[r\frac{d^2\theta}{dt^2} + \underbrace{2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)}_{\text{Coriolis acceleration.}}\right]\hat{\theta}}$$