Last Time: dynamical variables needed for describing superstrings

$\psi^I_{\alpha=1,2}$

$$\psi^I_1 = \Psi^I_1(\tau - \sigma)$$
$$\psi^I_2 = \Psi^I_2(\tau + \sigma)$$

BCs: $\psi^I_1(\tau, \sigma_*)\delta\psi^I_1(\tau, \sigma_*) - \psi^I_2(\tau, \sigma_*)\delta\psi^I_2(\tau, \sigma_*) = 0$

Suppose: $\psi^I_1(\tau, 0) = 0$ then $\Psi^I_1(\tau) = 0$ by $\psi^I_1 = \Psi^I(\tau - \sigma)$. Bad! Instead relate ψ_1 and ψ_2, assemble full spin or $\Psi^I(\tau, \sigma)$

$$\Psi_I(\tau, \sigma) = \begin{cases} \psi^I_1(\tau, \sigma) & \sigma \in [0, \pi] \\ \psi^I_2(\tau, -\sigma) & \sigma \in [-\pi, 0] \end{cases}$$

Ψ^I continuous at $\sigma = 0$ because of BC $\psi^1_1(\tau, \pi) = \pm\psi^I_2(\tau, \pi)$.

Either periodic or antiperiodic. Take periodic and get Rammond. (Actually more complicated). Take antiperiodic and get Neven Schwarz BCS (2 years after Rammond)

NS BC: $\Psi^I(\tau, \pi) = -\Psi^I(\tau, -\pi)$. $\Psi^I(\tau, \sigma) = \sum_{r \in Z + \frac{1}{2}} b^I_r e^{(-ir(\tau - \sigma))}$.

Creation Operations: $b^I_{-5/2}, b^I_{-3/2}, b^I_{-1/2}$
Destruction Operations: $b^I_{5/2}, b^I_{3/2}, b^I_{1/2}$

Ψ^I is anticommutative, all b^I_r operations anticommutative.

$$b^I_r, b^J_s = \delta_{r+s,0}\delta^{IJ}$$

NS State:

$$\left[\prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha^I_{-n})^{\lambda_{n,I}}\right]\left[\prod_{J=2}^{9}\prod_{r=\frac{1}{2},\frac{3}{2},\frac{5}{2}}(b^I_{-r})^{\rho_{r,I}}\right] \cdot |NS\rangle \otimes p^+, \vec{p}_\tau$$

$$\rho_{r,I} \in \{0, 1\}$$

Recall for open bosonic string, normal ordered:

$$M^2 = \frac{1}{\alpha'}\left(\frac{1}{2}\sum_{p\in Z} \alpha^I_{-p}\alpha^I_p\right)$$

Now:

$$M^2 = \frac{1}{\alpha'}\left(\frac{1}{2}\sum_{p\in Z} \alpha^I_{-p}\alpha^I_p + \frac{1}{2}\sum_{r\in Z+\frac{1}{2}} r b^I_{-r} b^I_r\right)$$

$$\begin{aligned}
\frac{1}{2}\sum_{r=-\frac{1}{2},-\frac{3}{2}} r b^I_{-r} b^I_r &= -\frac{1}{2}\sum_{r=\frac{1}{2},\frac{3}{2}} r b^I_r b^I_{-r} \\
&= \frac{1}{2}\sum_r r b^I_{-r} b^I_r \\
&= -\frac{1}{2}(D-2)\underbrace{\left(\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots\right)}_{\frac{1}{12}} \\
&= -\frac{1}{2}(D-2)\frac{1}{2}\frac{1}{12} \\
&= -\frac{1}{48}(D-2)
\end{aligned}$$

For boson, $a_B = -\frac{1}{24}$.

In open bosonic string, $M^2 = \frac{1}{\alpha'}(\dots + 1)$ where $a = -1$ but 24 contributions so $a_B = -\frac{1}{24}$.

Here, $a_{NS} = -\frac{1}{48}$ for antiperiodic fermion.

$$M^2 = \frac{1}{\alpha}(\sum_{p=1}^{\infty} \alpha^I_{-p}\alpha^I_p + \sum_{r=\frac{1}{2},\frac{3}{2}} r b^I_{-r} b^I_r + (D-2)(a_B + a_{NS}))$$

$$M^2 = \frac{1}{\alpha'}(N_{tot}^{\perp} - \frac{1}{2})$$

Add text here.

In early 1970s, confusion over whether these are bosons or fermions. It'll turn out that these are photons.

Count states of a given $N^{\perp}$

Given:

$$a_1^+ : f(x) = \sum_{n=0}^{\infty} \underbrace{a(n)}_{\text{number of states with } N^{\perp}=n} x^n$$

$|0N^{\perp} = 0$, $a_1^+|0N^{\perp} = 1$, $(a_1^+)^2|0N^{\perp} = 2$

$$f_1(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

Given:

$$a_2^+ : f_2(x) = 1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2}$$

$|0N^{\perp} = 0$, $a_2^+|0N^{\perp} = 2$, $(a_2^+)^2|0N^{\perp} = 4$

$$f_1(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

Given:

$$a_1^+, a_2^+ : f_1(x)f_2(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2}$$

Now can do full open bosonic string with $a_1^+, a_2^+, a_3^+, \ldots$

Generating Function:

$$f_{os} = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)}$$

$$f_{os} = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)} = \sum_{n=0}^{\infty} \underbrace{p(n)}_{\text{partitions of } n} x^n$$

Partitions of 4: $(\{4\}, \{3,1\}, \{2,2\}, \{2,1,1\}, \{1,1,1,1\})$ = number of ways to get $N^{\perp} = 4$

$$\boxed{\ln p(N) \approx 2\pi\sqrt{\frac{N}{6}}}$$

$$\boxed{p(N) \approx \frac{1}{4N\sqrt{3}} exp(2\pi\sqrt{\frac{N}{6}})} M$$

For full open string:

$$f_{os} = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}}$$

This gives you degeneracy of any level of open string.

Given b_1^+: $f_1(x) = 1 + x$
Given b_2^+: $f_2(x) = 1 + x^2$
Given b_1^+, b_2^+: $f_{12}(x) = f_1(x) f_2(x) = (1+x)(1+x^2)$
Given $b_{\frac{1}{2}}^+$: $f_{\frac{1}{2}}(x) = 1 + \sqrt{x}$

$$f_{NS}(x) = \sum_r \underbrace{a(r)}_{\#\text{ of states with } \alpha' M^2 = r} x^r = 1^{-\frac{1}{2}} + 8 \cdot x^0 + 36 \cdot x^{\frac{1}{2}} + (\#) x^1$$

$$= \frac{1}{\sqrt{x}} \prod_{n=1}^{\infty} \left(\frac{1 + x^{n-1}}{1 - x^n} \right)^8$$

Ramond: $\Psi^I(\tau, \sigma) = \sum d_n^I exp(-in(\tau - \sigma))$. $d_m^I, d_n^I = \delta_{m+n,0}\delta^{IJ}$.

$d_0^I \to 8$: 4 creation and 4 destruction $= \xi_1, \xi_2, \xi_3, \xi_4$

Vacuum State: $|0\rangle$

$|0\rangle$: 1
$\xi^I \xi^J |0\rangle$: 6
$\xi_1 \xi_2 \xi_3 \xi_4 |0\rangle$: 1

This yields 8, $|R_1^a\rangle$.

$\xi^I |0\rangle$: 4
$\xi^I \xi^J \xi^K |0\rangle$: 4

This yields 8, $|R_2^a\rangle$.

$8+8 = 16$ gound states. Total set of vacua states: $\left|R^A\right\rangle$, $A = 1 \ldots 16$, split into 2 types.

Ramond mass formula:

$$M^2 = \frac{1}{\alpha'}(\sum_{p=1}^{\infty} \alpha_{-p}^I \alpha_p^I + \sum_{n=1}^{\infty} n d_{-n}^I d_n^I)$$

Substraction constant is equal to zero since $a_R = \frac{1}{24}$

$\alpha' M^2 = 0$	$\|R_1^a\rangle$	$\|R_2^a\rangle$
$\alpha' M^2 = 1$	$\alpha_{-1}^I \|R_1^a\rangle$,$d_{-1}^I \|R_2^a\rangle$	$\alpha_{-1}^I \|R_2^a\rangle$,$d_{-1}^I \|R_1^a\rangle$

Why is this supersymmetry? Left and right columns have opposite fermionic states. Don't know if R_1^a is a boston or a fermion, but know R_2^a is the opposite.

No bosons that look like $|R_1^a\rangle$ in real world

Partition function in Raman Sector

$$f_R(x) = 16 \prod_{n=1}^{\infty} \left(\frac{1+x^n}{1-x^n} \right)^8$$

To get supersymmtry, throw out half of the states from each sector and put them together.

$$f_{NS}^{\text{truncated}} = \frac{1}{2\sqrt{x}}\left[\prod_{n=1}^{\infty}\left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\right)^8 - \prod_{n=1}^{\infty}\left(\frac{1-x^{n-\frac{1}{2}}}{1-x^n}\right)^8\right]$$

Anything with an odd number of fermions will change the sign.

$$f_{NS} \stackrel{?}{=} 8\prod_{n=1}^{\infty}\left(\frac{1+x^n}{1-x^n}\right)^8$$

Do we or don't we have supersymmetry?

1829: German Mathematician Jacoby wrote treatise on elliptic function with this identity, labelled "a very strange identity". Critical dimension $D = 10$ for supersymmetry.