Lecture 20 - Topics

- Closed Strings

Recall: For closed strings in light cone gauge:

$$\sigma \approx \sigma + 2\pi$$
$$x^+ = \alpha' p^+ \tau$$
$$\mathcal{P}^{\tau +}\text{constant}$$
$$\dot{X} \pm X^{'-} = \frac{1}{\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X'^I)^2$$
$$H = \alpha' p^+ p^-$$
$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu$$

Open Strings:

$$[X^I(\tau,\sigma), \mathcal{P}^{\tau,J}(\tau,\sigma')] = i\eta^{IJ}\delta(\sigma-\sigma')$$
$$\dot{X}^I \pm X'^I = \sqrt{2\alpha'}\sum \alpha_n e^{(-in(\tau\pm\sigma))}$$
$$[\alpha^I_m, \alpha^I_n] = m\delta_{m+n,0}\delta^{IJ}$$

Graviton States:

$$\xi_{IJ} a^{IJ+}_{p^+,p_\tau} |\Omega\rangle$$
$$\xi_{IJ} = \xi_{JI} = \xi^I_J = 0$$

Solve and Find Mode Expansion of Closed Strings

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)$$

Left and Right (X^μ_L and X^μ_R) both solve wave equation, as goes their sum. Let:

$$u = \tau+\sigma, v = \tau-\sigma$$

$$x^\mu(\tau, \sigma + 2\pi) = x^\mu(\tau, \sigma)$$

True $\sigma \approx \sigma + 2\pi$ *except* when the wold has a compact dimension and x goes aroudn a circle - even though back at same σ after 2π, at a different x coordinate.

$$X_L(u) + X_R(v) = X_L(u + 2\pi) + X_R(v - 2\pi)$$

$$\boxed{X_R(v) - X_R(v - 2\pi) = X_L(u + 2\pi) - X_L(u)}$$

This is the periodicity condition. X_L and X_R are independent variables. X_L' and X_R' are periodic.

$$X_L'^\mu(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in Z} \overline{\alpha_n}^\mu e^{(-inu)}$$

$$X_R'^\mu(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in Z} \alpha_n^\mu e^{(-inv)}$$

Have 2 independent sets of oscillators. *Not* related to the open string oscillators.

$$X_L^\mu(u) = \frac{1}{2} X_{L0}^\mu + \sqrt{\frac{\alpha'}{2}} \overline{\alpha_0}^\mu u + i\sqrt{\ldots}$$

$$X_R^\mu(v) = \frac{1}{2} X_{R0}^\mu + \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu v + \ldots$$

All terms in X_L^μ and X_R^μ have e^{-inu} component except first two terms of each.

Periodicity Condition:

$$\boxed{\alpha_0^\mu = \overline{\alpha}_0^\mu \forall \mu}$$

$$X^\mu(\tau, \sigma) = \frac{1}{2}(X_{L0}^\mu + X_{R0}^\mu) + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum \ldots$$

Let: $x_0^\mu = \frac{1}{2}(X_{L0}^\mu + X_{R0}^\mu)$

Momentum of String:

$$\begin{aligned}
p^\mu &= \int_0^{2\pi} \mathcal{P}^{\tau\mu} d\sigma \\
&= \frac{1}{2\pi\alpha'} \int_0^{2\pi} \frac{\partial x^\mu}{\partial \tau} d\sigma \\
&= \frac{1}{2\pi\alpha'} \sqrt{2\alpha'} \alpha_0^\mu (2\pi) \\
&= \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu
\end{aligned}$$

$$\boxed{\dot{X}^\mu = X_L^{\mu\prime}(\tau+\sigma) + X_R^{\mu\prime}(\tau-\sigma)}$$

$$\boxed{X^{\mu\prime} = X_L^{\mu\prime}(\tau+\sigma) - X_R^{\mu\prime}(\tau-\sigma)}$$

$$\boxed{\dot{X}^\mu + X^{\mu\prime} = 2X_L^{\mu\prime} = \sqrt{2\alpha'} \sum n \in Z \overline{\alpha}_n^\mu e^{(-in(\tau+\sigma))}}$$

$$\boxed{\dot{X}^\mu - X^{\mu\prime} = 2X_R^{\mu\prime} = \sqrt{2\alpha'} \sum n \in Z \alpha_n^\mu e^{(-in(\tau-\sigma))}}$$

$$(\dot{X}^I + X^{I\prime})^2 = 4\alpha' \sum \overline{L}_n^+ e^{(-in(\tau+\sigma))} \Rightarrow \overline{L}_n^\perp = \frac{1}{2} \sum_p \overline{\alpha}_p^I \overline{\alpha}_{n-p}^I$$

$$(\dot{X}^I - X^{I\prime})^2 = 4\alpha' \sum \overline{L}_n^+ e^{(-in(\tau-\sigma))} \Rightarrow \overline{L}_n^\perp = \frac{1}{2} \sum_p \alpha_p^I \alpha_{n-p}^I$$

$$\begin{aligned}
\dot{X}^- + X^{-\prime} &= \frac{1}{\alpha'} \frac{1}{2p^+} 4\alpha' \sum L_n^+ e^{(-in(\tau+\sigma))} \\
&= \frac{2}{p^+} \sum \overline{L}_n^+ e^{(-in(\tau+\sigma))} \\
&= \sqrt{2\alpha'} \sum \overline{\alpha}_n^- e^{(-in(\tau+\sigma))} \dot{X}^- - X^{-\prime} = \frac{2}{p^+} \sum L_n^\perp e^{(-in(\tau-\sigma))}
\end{aligned}$$

$$\boxed{\sqrt{2\alpha'}\overline{\alpha}_n^- = \frac{2}{p^+}\overline{L}_n^\perp}$$

$$\boxed{\sqrt{2\alpha'}\alpha_n^- = \frac{2}{p^+}L_n^\perp}$$

$|\overline{L}_0^\perp = L_o^\perp|$ constraint on state space of theory.

Differences between closed and open string. Hilber space: can't just double everything for closed strings. X_0 doesn't double and momentum doesn't double. $\overline{L}_0^\perp = L_0^\perp$ constant.

$L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + N^\perp \leftarrow$ number operator $= \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I$
$L_0^\perp = \frac{\alpha'}{4}p^Ip^I + N^\perp$ (For open strings, did not have factor $\frac{1}{4}$
$L_0^\perp = \frac{\alpha'}{4}p^Ip^I + \overline{N}^\perp$, $\overline{N}^\perp = \sum_{p=1}\overline{\alpha}_{-p}^I\overline{\alpha}_p^I$

$$L_0^\perp = \overline{L}_0^\perp \Rightarrow N^\perp = \overline{N}^\perp$$

Recall:

$$\sqrt{2\alpha'}\overline{\alpha}_n^- = \frac{2}{p^+}(\overline{L}_n^\perp - 1)$$

$$\sqrt{2\alpha'}\alpha_n^- = \frac{2}{p^+}(L_n^\perp - 1)$$

$$\boxed{\sqrt{2\alpha'}\alpha_0^- = \frac{1}{p^+}(L_0^\perp + \overline{L}_0^+ - 2) = \alpha' p^-}$$

$$\boxed{H = \alpha' p^+ p^- = L_0^\perp + \overline{L}_0^\perp - 2}$$

$$\boxed{M^2 = \left(\frac{2}{\alpha'}N^\perp + \overline{N}^\perp - 2\right)}$$

Recall open string: $M^2 = \frac{1}{\alpha'}(-1 + N^\perp)$

$$[L_0^\perp + \overline{L}_0^\perp, X^I(\tau,\sigma)] = -i\frac{\partial x^I}{\partial\tau}$$

$$[L_0^\perp - L_0^\perp, X^I(\tau,\sigma)] = i\frac{\partial x^I}{\partial \tau}$$

$$\begin{aligned} X^I(\tau,\sigma+) &= X^I(\tau,\sigma) + \frac{\partial X^I}{\partial \sigma} \\ &= X^I(\tau,\sigma) + [-i\epsilon(L_0^\perp - L_0^\perp), X^I(\tau,\sigma)] \end{aligned}$$

$$\prod_{n=1}^{\infty}\prod_{I=2}^{25}(a_n^{I+})^{\lambda_{n,I}} \prod_{m=1}^{\infty}\prod_{J=2}^{25}{}^{\lambda_{m,J}} \left|p^+, p_\tau\right\rangle$$
$$N^\perp = \overline{N}^\perp$$

Ground state: $N^\perp = \overline{N}^\perp = 0$, $|p^+, p_\tau\rangle$, $M^2 = -\frac{4}{\alpha'}$. Close string tachyon. Not well understood.

Next state: $M = R_{IJ} a_1^{I+} \overline{a}_1^{J+} |p^+, p_\tau\rangle$, $N^\perp = \overline{N}^\perp = 1$, $M^2 = \frac{2}{\alpha'}(1+1-2) = 0$.

We have a $(D-2)$x$(D-2)$ matrix. Any matrix can be split into symmetric and antisymmetric: $R_{IJ} = S_{IJ} + A_{IJ}$

$$\begin{aligned} S &= \frac{R + R^T}{2} \\ A &= \frac{R - R^T}{2} \end{aligned}$$

$$\begin{aligned} R_{IJ} &= (S_{IJ} - \frac{1}{D-2}\delta_{IJ} S_K^K) + (\frac{1}{D-2}\delta_{IJ} S_K^K + A_{IJ}) \\ &= \hat{S}_{IJ} + S'\delta_{IJ} + A_{IJ} \end{aligned}$$

$$M = \hat{S}_{IJ} a_1^{I+} \overline{a}_1^{J+} \left|p^+, p_\tau\right\rangle + A_{IJ} a_1^{I+} \overline{a}_1^{J+} \left|p^+, p_\tau\right\rangle + S' a_1^{I+} \overline{a}_1^{I+} \left|p^+, p_\tau\right\rangle$$

$\hat{S}_{IJ} \leftrightarrow \xi_{IJ} a_{p^+}^{IJ} p_\tau |\Omega\rangle$ graviton states

A_{IJ}: "Kalbra-mon" states
S' just one single state. No Lorentz index. Massless scalar. A real troublemaker. Called dilation scalar. Tells us how strong strings interact.