VII The Bose Gas.

① Photons:

ⓐ Photons are bosons

ⓑ No particle number conservation → Cannonical Ensemble

no Grand cannonical ensemble

* How to label photon state:

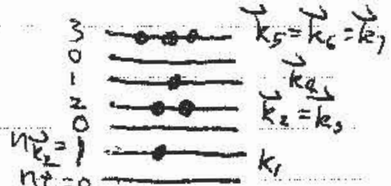
one boson: $|\vec{k}_1\rangle$, $\vec{k}_1 = \frac{2\pi}{L}(n_x, n_y, n_z)$

~~two~~ bosons: $|\vec{k}_1, \vec{k}_2\rangle$

$= |\vec{k}_2, \vec{k}_1\rangle$ $\quad \vec{k}_1 = \vec{k}_2$ OK

↑ identical particales

N- bosons $|\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_N\rangle$



Another label $|n_{\vec{k}_1}\, n_{\vec{k}_2} \cdots\rangle$

$n_{\vec{k}} = 0, 1, 2, \cdots, \infty$ independently

$[n_{\vec{k}} = 0, 1 \quad (\text{fermion})]$

$$Q = \sum_{n_{k_1}=0,1,\cdots} \sum_{n_{k_2}=0,1\cdots} e^{-\beta \sum_k n_k \epsilon_k}$$

$$= \prod_k \sum_{n_k=0,1,\cdots} e^{-\beta n_k \epsilon_k}$$

↑ Prob. for $\vec{k}$-state to have n_k photons

$$= \prod_k \frac{1}{1-e^{-\beta\epsilon_k}}$$

Free energy:

$$A = -k_B T \ln Q$$

two different polarization

$$= + k_B T \sum_k \ln(1 - e^{-\beta \epsilon_{\vec{k}}}) \times 2$$

The Total system is a sum of independent sub systems: each k-level = one subsystem

The state of each sub system in labeled by $n_k = 0, 1, \cdots$ energy $= n_k \epsilon_k$

$$Q_k = \sum_{n_k} e^{-\beta n_k \epsilon_k} = \frac{1}{1 - e^{-\beta \epsilon_{\vec{k}}}}$$

$$A_k = k_B T \ln(1 - e^{-\beta \epsilon_{\vec{k}}})$$

total Free energy

$$A = \sum_k A_k = k_B T \sum_k \ln(1 - e^{-\beta \epsilon_{\vec{k}}})$$

Each subsystem = a harmonic oscillator

with $\hbar\omega = \epsilon_k$

energy $E = n\hbar\omega + \frac{1}{2}\hbar\omega$

A photon system = A collection of oscillators labeled by $\vec{k}$ with $\hbar\omega_k = \epsilon_k$

Prob. for level-$\vec{k}$ to have n_k photons

$$P(n_k) = \frac{e^{-\beta n_k \epsilon_k}}{\sum_{n_k} e^{-\beta n_k \epsilon_k}} = e^{-\beta \epsilon_k n_k}(1 - e^{-\beta \epsilon_k})$$

Average # of photon on level-$\vec{k}$

$$\langle n_{\vec{k}} \rangle = \sum_{n_k} P(n_k)\, n_k$$

$$= (1 - e^{-\beta\epsilon_k}) \sum_{n_k} n_k e^{-\beta n_k \epsilon_k}$$

$$= (1 - e^{-\beta\epsilon_k}) (-) \frac{\partial}{\partial(\beta\epsilon_k)} \sum_{n_k} e^{-(\beta\epsilon_k) n_k}$$

$$= (1 - e^{-\beta\epsilon_k}) (-) \frac{\partial}{\partial(\beta\epsilon_k)} \left(\frac{1}{1 - e^{-\beta\epsilon_k}} \right)$$

$$= (1 - e^{-\beta\epsilon_k}) \frac{e^{-\beta\epsilon_k}}{(1 - e^{-\beta\epsilon_k})^2}$$

$$\boxed{\langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta\epsilon_k} - 1}} \quad \text{boson} \qquad \boxed{\langle n_k \rangle = \frac{1}{e^{\beta\epsilon_k} + 1}} \quad \text{fermion}$$

# of photons on level-$\vec{k}$ with a given polarization

Each k-level has a volume $\left(\frac{2\pi}{L}\right)^3$ in k-space

$$\sum_k \Leftrightarrow V \int \frac{d^3k}{(2\pi)^3}$$

Total # of photons

$$\boxed{\epsilon_{\vec{k}} = \hbar\omega_{\vec{k}} = \hbar|\vec{k}|c} \qquad \boxed{\omega = ck}$$

$$N = 2 \sum_k n_k$$

$$= 2V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\epsilon_k} - 1}$$

$$n = \frac{2}{(2\pi)^3} \int_0^\infty dk \frac{4\pi k^2}{e^{\beta\hbar c k} - 1} = \frac{1}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^2}{e^{\beta\omega\hbar} - 1}$$

$\uparrow$ photon density

$$= K \left(\frac{k_B T}{\hbar c} \right)^3$$

$$K = \frac{1}{\pi^2} \int_0^\infty dx \frac{x^2}{e^x - 1} = 0.23\ldots$$

Physical picture:

typical energy of a photon $\sim k_B T$

typical wave length: $k \sim \frac{k_B T}{\hbar c}$

$\lambda \sim \frac{\hbar c}{k_B T}$ ← thermal wave length.

one photon per λ^3

$\Rightarrow$ density $n \approx \frac{1}{\lambda^3} \approx \left(\frac{k_B T}{\hbar c}\right)^3$

* Energy density $U \sim n \cdot k_B T \sim \left(\frac{k_B T}{\hbar c}\right)^3 k_B T$

$\propto T^4$

Stefan's law

$$U = 2 \sum_k \epsilon_k n_k$$

$$= \frac{8\pi c V \hbar}{(2\pi)^3} \int_0^\infty dk \frac{k^3}{e^{\beta c \hbar k} - 1}$$

$$= \frac{V\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta \omega \hbar} - 1}$$

Classical:

$$U = 2 \sum_k k_B T$$

$$= 2 V k_B T \int \frac{d^3 k}{(2\pi)^3}$$

$$= \frac{2 V k_B T}{(2\pi)^3} \int 4\pi k^2 dk = \infty$$

$$\frac{U}{V} = \sigma T^4 \qquad \sigma = \frac{\pi^2 k_B^4}{15 (\hbar c)^3}$$ Stefan's const.

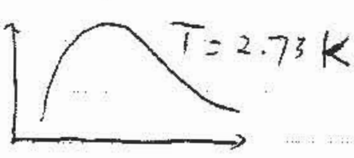
$$\frac{U}{V} = \int_0^\infty d\omega \, u(\omega, T)$$

$d\omega \, u(\omega, T)$ = energy density with frequency between ω & $\omega + d\omega$

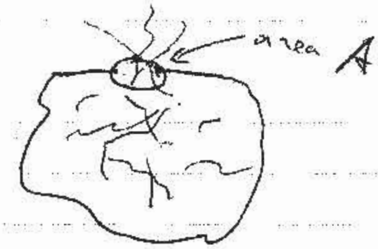
$u(\omega, T)$ spectral density

$$\boxed{u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta \hbar \omega} - 1}}$$

Planck distribution



★ Black body radiation

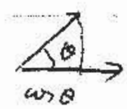


Total power

$$\frac{W}{A} = \frac{U}{V}\,\frac{1}{2}\,c \cdot \left(\underbrace{\frac{\int_0^{\frac{\pi}{2}} d\theta \,\sin\theta \cdot \cos\theta \;(2\pi)}{\int_0^{\frac{\pi}{2}} d\theta \,\sin\theta \;(2\pi)}}_{\langle \cos\theta \rangle} \right.$$

$= \frac{1}{2}$

$= 1$



$$\int_0^1 d(\cos\theta)\,\cos\theta = \frac{1}{2}$$

$$= \frac{1}{4}\frac{U}{V}\,c$$

$$\boxed{\frac{W}{A} = \frac{c}{4}\,\sigma T^4}$$ Total power per area

power spectrum $\frac{w}{A} = \frac{c}{4}\,u(\omega, T)$

$$= \frac{c}{4}\,\frac{\hbar}{\pi^2 c^3}\,\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

★ Pressure:

$$P = -\frac{\partial A}{\partial V} = -2\sum_k \frac{\frac{\partial \epsilon_k}{\partial V}\, e^{-\beta\epsilon_k}}{1-e^{-\beta\epsilon_k}}$$

$\epsilon_k \propto \frac{1}{V^{1/3}}$

$\frac{\partial \epsilon_k}{\partial V} = \frac{-1}{3}\,\frac{\epsilon_k}{V}$

$$= \frac{1}{3V}\; 2\sum_k \frac{\epsilon_k e^{-\beta\epsilon_k}}{1-e^{-\beta\epsilon_k}} = \frac{1}{3V}\,U$$

$$\Rightarrow \quad \boxed{P = \frac{1}{3}\,\frac{U}{V}}$$ radiation pressure.

② Phonons: vibration of lattice

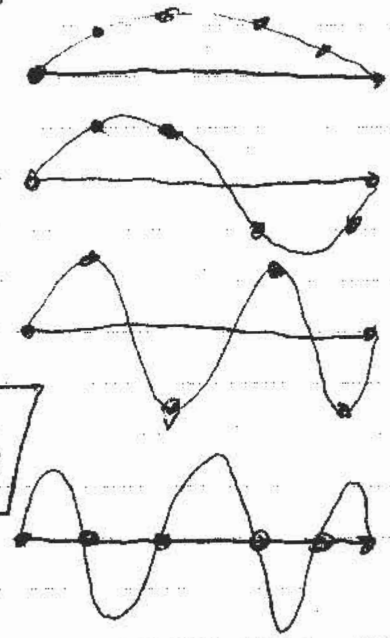
Phonons are just like photons

Three differences:

① Three polarizations (3D)

② velocity $v \ll c$

③ Upper bond of $\vec{k}$



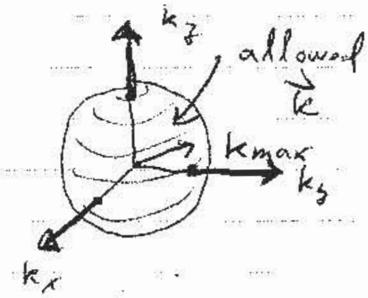
$$\boxed{\#\text{ of } \vec{k}\text{-levels} = \#\text{ of lattice site}}$$

$$\#\text{ of } \vec{k}\text{-levels} = \sum_{\vec{k}} 1$$

$$= V \int \frac{d^3k}{(2\pi)^3}\, 1$$

$$= V \frac{4\pi}{3} k_{max}^3 = N_{site}$$

$$\Rightarrow \boxed{k_{max} = \left(\frac{3 N_{site}}{V}\right)^{1/3}}$$



Debye-mode

Ⓐ $\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $|\vec{k}| < k_{max}$

Ⓑ $\epsilon_k = \hbar v |\vec{k}|$

Einstein model

Ⓐ allowed k-levels $= N_{site}$ (or $|\vec{k}| < k_{max}$)

Ⓑ $\epsilon_k = \epsilon_0$

Density of states in Debye model

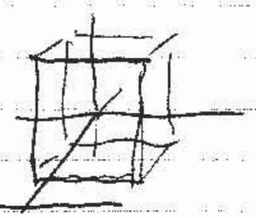
$$D(\epsilon) = 3V \int \frac{d^3k}{(2\pi)^3}\, \delta(\epsilon - \hbar v k)$$

$$= 3 \frac{V}{2\pi^2} \int dk\; k^2\, \delta(\epsilon - \hbar v k)$$

$$= \frac{3}{2\pi^2} \frac{V}{(\hbar v)^3} \int_0^{\epsilon_{max}} d\epsilon'\, \delta(\epsilon - \epsilon')\, (\epsilon')^2$$

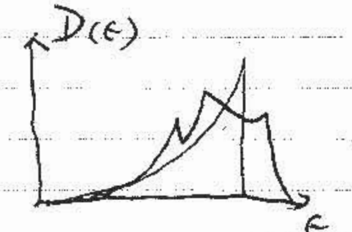
$$= \begin{cases} \frac{3}{2\pi} \frac{V}{(\hbar v)^3}\, \epsilon^2 & \epsilon < \epsilon_{max} = \hbar v k_{max} \\ 0 & \epsilon > \epsilon_{max} \end{cases}$$

real material



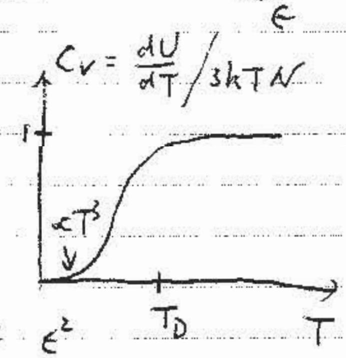
(A) allowed k-levels

(B) $\epsilon_k = v\sqrt{\sin^2 k_x a + \sin^2 k_y a + \sin^2 k_z a}$



* Debye specific heat

internal energy



$$\frac{U}{N} = \int d\epsilon\, D(\epsilon) \frac{\epsilon}{e^{\beta\epsilon} - 1} \Big/ \overbrace{\frac{1}{3}\int d\epsilon\, D(\epsilon)}^{\to 3N}$$

$$= 3 \int_0^{\epsilon_{max}} d\epsilon \frac{\epsilon^3}{e^{\beta\epsilon} - 1} \Big/ \int_0^{\epsilon_{max}} d\epsilon\, \epsilon^2$$

$$= 9\, \epsilon_{max}^{-3} \int_0^{\epsilon_{max}} d\epsilon \frac{\epsilon^3}{e^{\beta\epsilon} - 1} \qquad t = \beta\epsilon$$

$$= 3 k_B T\, D(u) \quad \boxed{u = \frac{T_D}{T}} \qquad T_D = \frac{\epsilon_{max}}{k_B}$$

$$D(u) = \frac{3}{u^3} \int_0^u dt \frac{t^3}{e^t - 1} = \begin{cases} 1 - \frac{3}{8}u \cdots & (u \ll 1) \; T \gg T_D \\ \frac{\pi^4}{5u^3} \cdots & (u \gg 1) \; T \ll T_D \end{cases}$$

Debye temperature

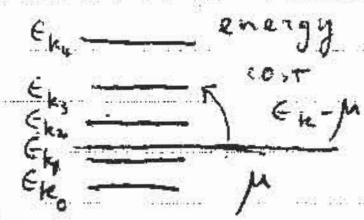
③ Boson Condensation

Boson system with conserved boson number.

Grand partition function

$$Q_G = \sum_{\{n_k\}} e^{-\beta\left(\sum_k n_k \epsilon_k - \mu N\right)}_{\,\underset{N=\sum_k n_k}{}}$$

$$= \prod_k \underbrace{\sum_{n_k} e^{-\beta n_k(\epsilon_k - \mu)}}_{\substack{\text{partition function} \\ \text{of one oscillator} \\ \text{with } \boxed{\hbar\omega = \epsilon_k - \mu}}} = \prod_k \frac{1}{1 - e^{\beta(\epsilon_k - \mu)}}$$



$e^{-\beta n_k(\epsilon_k - \mu)} \propto$ Prob. for the oscillator to in the n_k^{th} state

$=$ Prob. for the level-k to have n_k bosons.

$$= e^{-\beta n_k(\epsilon_k - \mu)}\left(1 - e^{-\beta(\epsilon_k - \mu)}\right)$$

$$\boxed{\langle n_k \rangle = \frac{1}{e^{-\beta(\epsilon_k - \mu)} - 1}}$$ bose distribution

☆ Occupation numbers of N-boson system.

$$N = \sum_k n_k \qquad \leftarrow \mu \text{ is such that there are } N \text{ bosons}$$

$$= V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

$$T \to \infty \qquad \mu \to -\infty \qquad n_k \to 0$$

$$N = V \int \frac{d^3k}{(2\pi)^3}\, e^{-\beta(\epsilon_k - \mu)} \qquad \boxed{\lambda = \sqrt{2\pi\hbar^2/mk_BT}}$$

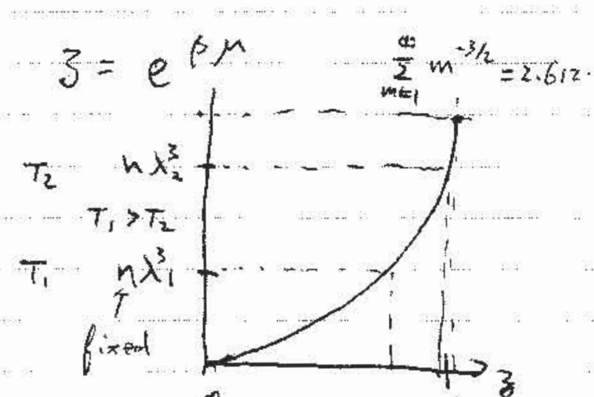
$$= V e^{\beta\mu} \lambda^{-3} \qquad n\lambda^3 = e^{\beta\mu} = z$$

In general

$$\text{boson-density} = n = \int \frac{d^3k}{(2\pi)^3} \sum_{m=1}^{\infty} e^{-m\beta(\epsilon_k - \mu)}$$

$$= \sum_{m=1}^{\infty} e^{m\beta\mu} \underbrace{\int \frac{d^3k}{(2\pi)^3}\, e^{-m\beta\epsilon_k}}_{\frac{1}{m^{3/2}} \underbrace{\int \frac{d^3k}{(2\pi)^3} e^{-\beta\epsilon_k}}_{\lambda^{-3}}}$$

$$\Rightarrow \boxed{\begin{aligned} n\lambda^3 &= \sum_{m=1}^{\infty} \frac{z^m}{m^{3/2}} \\ &= g_{3/2}(z) \end{aligned}} \qquad z = e^{\beta\mu}$$



But μ (or z) has no solution if

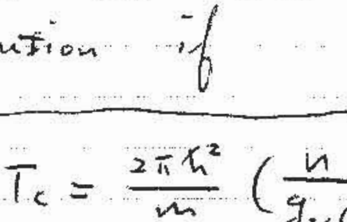
$n\lambda^3 > g_{3/2}(1)$ or $k_B T < \boxed{k_B T_c = \frac{2\pi\hbar^2}{m}\left(\frac{n}{g_{3/2}(1)}\right)^{2/3}}$

★ What happens: or $\boxed{n > n_c = g_{3/2}(1)\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}}$

When $\mu = 0$

the excited levels

contain most bosons



As we add more bosons

we cannot increase μ further.

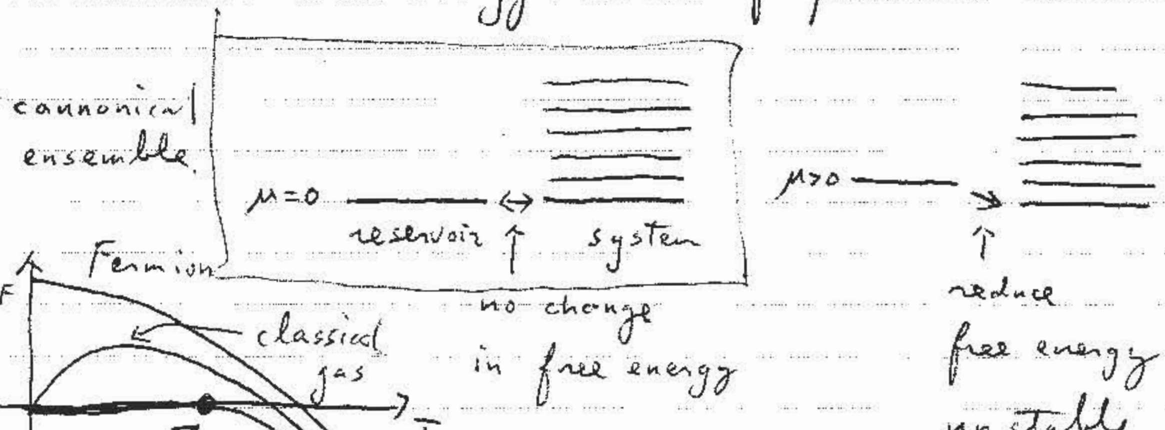
But when $\mu=0$, the $k=0$ level

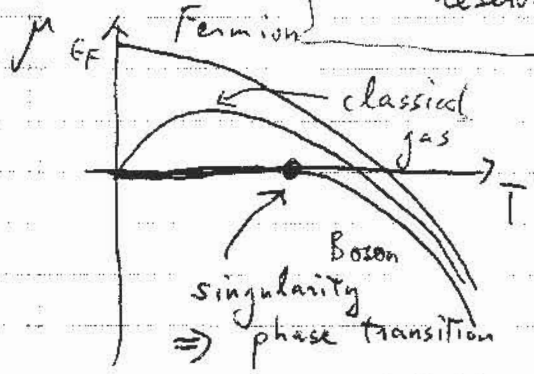
can have any numbers of boson.

(adding boson to $k=0$ level cost

energy $\underline{\underline{\mu}}$ for each boson)

↳ energy level of particle reservoir



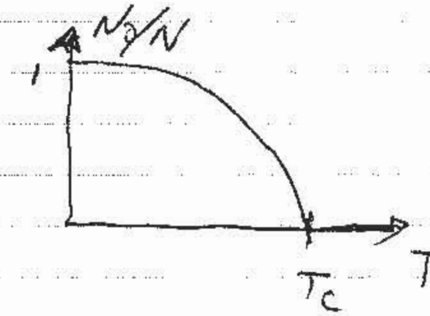


In the condensed state $\left(\begin{matrix} T < T_c \\ \text{or } n > n_c \end{matrix}\right)$

$$N = N_0 + \frac{V}{\lambda^3} g_{3/2}(1)$$

$$\Rightarrow \frac{N_0}{N} = 1 - \frac{g_{3/2}(1)}{n\lambda^3}$$

$$= 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

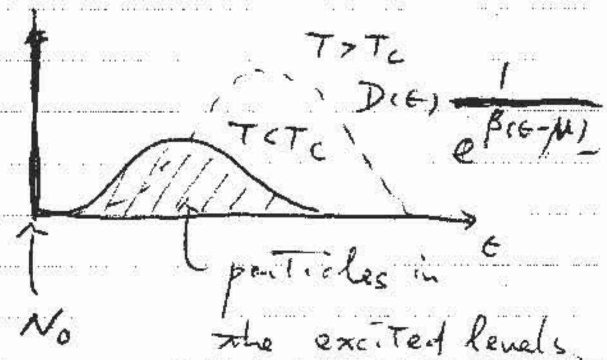


* Order parameter

condensed phase $\frac{N_0}{N} \neq 0$

gas phase $\frac{N_0}{N} = 0$.

So $\frac{N_0}{N}$ is an order parameter

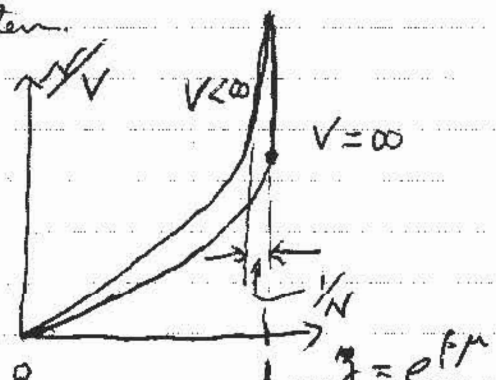


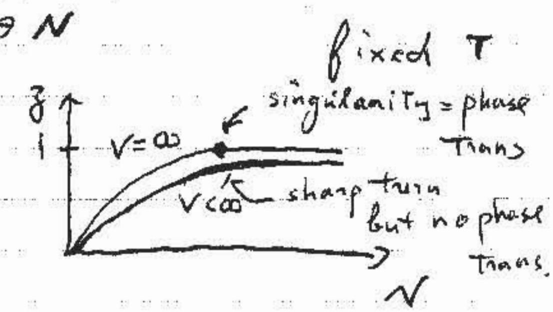
* No phase transition in finite system

$$N = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} = \sum_k \frac{1}{z e^{\beta \epsilon_k} - 1}$$

$$N_0 = \frac{1}{z - 1}$$

$$1 - z : \frac{2}{N} \to \frac{1}{N} \qquad N_0 = \frac{N}{2} \to N$$





★ Equation of state.

Thermal potential.

$$\Omega = -k_B T \ln Q_G = +k_B T \sum_k \ln\left[1 - e^{-\beta(\epsilon_k - \mu)}\right]$$

pressure

$$P = -\frac{\partial \Omega}{\partial V}\Big|_{\mu,T} = -k_B T \sum_k \frac{e^{-\beta(\epsilon_k-\mu)}}{1-e^{-\beta(\epsilon_k-\mu)}} \beta \frac{\partial \epsilon_k}{\partial V}$$

$$= \sum_k n_k \left(-\frac{\partial \epsilon_k}{\partial V}\right)$$

$$\epsilon_k = \frac{\hbar k^2}{2m} = * \frac{1}{V^{2/3}}$$

$$= \frac{2}{3V} \sum_k n_k \epsilon_k \Rightarrow \boxed{PV = \frac{2}{3} U}$$

$$\frac{\partial \epsilon_k}{\partial V} = -\frac{2}{3}\frac{\epsilon_k}{V}$$

$$U = \sum_k n_k \epsilon_k$$

$$\boxed{g_k(z) = \sum_{m=1}^{\infty} m^{-k} z^m}$$

$$\boxed{\frac{1}{z}\int_0^z dz\, g_k(z) = g_{k+1}(z)}$$

$$= V \int \frac{d^3k}{(2\pi)^3} \epsilon_k \underbrace{\sum_{m=1}^{\infty} z^m e^{-m\beta\epsilon_k}}_{\frac{1}{z^{-1}e^{\beta\epsilon_k}-1}}$$

$$= -V \frac{\partial}{\partial \beta}\Big|_z \int \frac{d^3k}{(2\pi)^3} \sum_{m=1}^{\infty} \frac{1}{m} z^m e^{-m\beta\epsilon_k}$$

$$\boxed{\begin{aligned}\int \frac{d^3k}{(2\pi)^3} e^{-\beta\epsilon_k} &= \lambda^{-3} \\ \int \frac{d^3k}{(2\pi)^3} e^{-m\beta\epsilon_k} &= m^{-3/2}\lambda^{-3}\end{aligned}}$$

$$= -V \frac{\partial}{\partial \beta}\Big|_z \sum_{m=1}^{\infty} \frac{1}{m} z^m m^{-3/2} \lambda^{-3}$$

$$\lambda = \sqrt{2\pi\hbar^2/m k_B T}$$

$$= -V \frac{\partial}{\partial \beta}\Big|_z \lambda^{-3} \sum_{m=1}^{\infty} \frac{1}{m^{5/2}} z^m$$

$$= -V g_{5/2}(z) \frac{\partial}{\partial \beta}\Big|_z \lambda^{-3}$$

$$\lambda^{-3} \propto T^{3/2} \propto \beta^{-3/2}$$

$$\frac{\partial}{\partial \beta}\lambda^{-3} = -\frac{3}{2}\lambda^{-3}/\beta$$

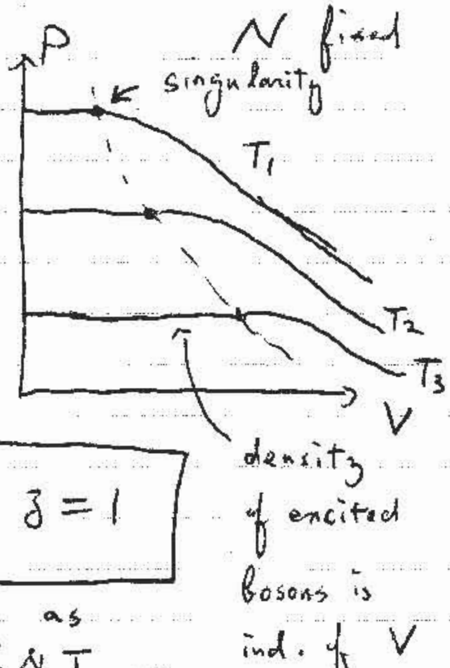
$$= \frac{3}{2} V \lambda^{-3} g_{5/2}(z)\, k_B T$$

$$\boxed{\frac{U}{V} = \frac{3}{2} k_B T \, \frac{g_{5/2}(z)}{\lambda^3}}$$

$$\boxed{P = \frac{2}{3}\frac{U}{V} = k_B T \, \frac{g_{5/2}(z)}{\lambda^3}}$$

$$\boxed{n\lambda^3 = g_{3/2}(z) = \frac{N}{V}\lambda^3 \quad \text{or} \quad z = 1}$$

↳ determine $z(n, T)$ as a function of $\frac{N}{V}$ & T.

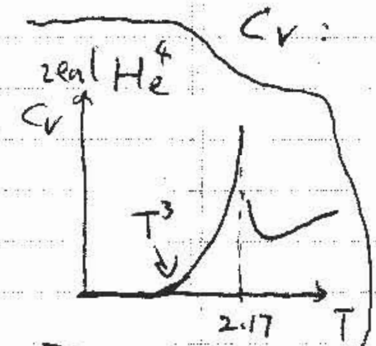


☆ Specific heat :

Condensed phase : $z = 1$

$$U = \frac{3}{2} k_B T \, \frac{g_{5/2}(1)}{\lambda^3} \propto T^{5/2}$$

$$C_V = \frac{\partial U}{\partial T}\Big|_{V,N} = \frac{5}{2}\cdot\frac{3}{2} k_B \frac{g_{5/2}(1)}{\lambda^3} V = \boxed{\frac{15}{4} k_B \frac{g_{5/2}(1)}{\lambda^3} V = C_V}$$



C_V : $T^{3/2}$ conserved free boson with $\epsilon_k \propto k^2$

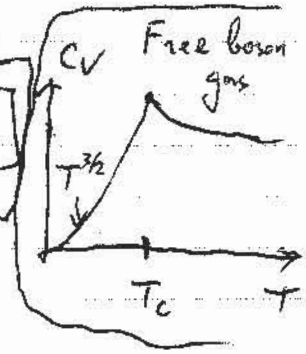
T conserved fermion with ϵ_k = anything

T^3 non-conserved free bosons with $\epsilon_k \propto k$

T^3 conserved interacting bosons with ϵ_k = anything

gas phase :

$$\boxed{C_V = k_B V \left[\frac{15}{4} \frac{g_{5/2}(z)}{\lambda^3} - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)} \right]}$$



$$g_{1/2}(1) = \infty$$

Boson condensation in ultracold atom system

Transition temperature

$$n\lambda^3 = g_{3/2}(1) \quad \Rightarrow \quad k_B T_c = \frac{2\pi\hbar^2}{m}\left(\frac{n}{g_{3/2}(1)}\right)^{2/3}$$

$\lambda = \sqrt{2\pi\hbar^2/mk_BT}$ $\qquad g_{3/2}(1) = 2.612\ldots$

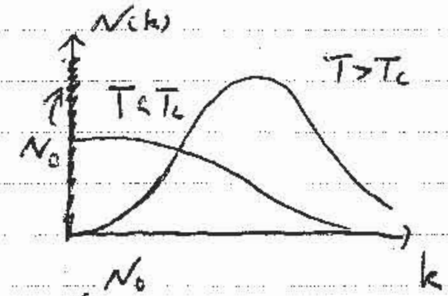
for Na atoms $\quad$ if $n = 10^{14}\ \mathrm{cm}^{-3}$
$Z=11,\ A=23$ $\qquad T_c = 1.5\,\mu K$

for He $\quad \rho = 0.12\ \mathrm{g\,cm}^{-3} \quad n = 1.8\times10^{22}\ \mathrm{cm}^{-3}$
$Z=2,\ A=4$ $\qquad T_c = 1.8K \quad$ actual $T_c = 2.17$

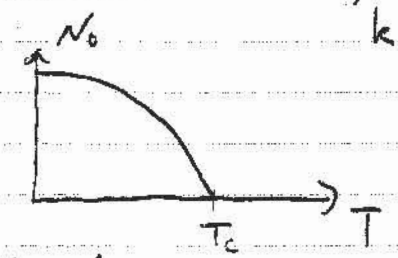
Momentum distribution

$N(k)\,dk$ = # of bosons with wave vector between k & $k+dk$

$$= V\,4\pi k^2\,\frac{1}{e^{\beta(\epsilon_k-\mu)}-1}$$



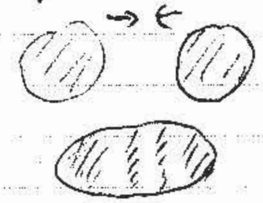
Condensation $N_0 = N \quad \frac{V}{\lambda^3}\,g_{3/2}(1)$



Detecting condensation

Time fly:

Interference :



④ Interacting bosons and superfluidity

$T=0$ All bosons condense.
All bosons are in the same state.

<u>Field Theory for free condensed bosons:</u>

☆ One boson S-eq:

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)}_{H}\psi}$$

total # of particle for state ψ

$$\int d^3x \; \psi^* \psi = 1$$

total energy for state ψ

$$E_1 = \int d^3x \; \psi^* H \psi$$
$$= \int d^3x \left(\frac{\hbar^2}{2m} \underbrace{|\nabla\psi|^2}_{|\partial_x\psi|^2+|\partial_y\psi|^2+|\partial_z\psi|^2} + U|\psi|^2\right)$$

☆ N-boson describe by the same wave fuction.

$$\boxed{\begin{aligned} &\int d^3x \, |\psi|^2 = N, \quad |\psi|^2 = \rho \quad \text{(boson density)} \\ &\int d^3x \left(\frac{\hbar^2}{2m}|\nabla\psi|^2 + U|\psi|^2\right) = E = NE_1 \end{aligned}}$$

Not quite right N-boson wave function $\Psi(\vec{x}_1, \cdots \vec{x}_N)$
$= \prod_i \psi(\vec{x}_i)$

* Ground state is given by ψ_0 that minimize E and satisfy $\int d^3x\, |\psi_0|^2 = N$

example: $U = 0 \quad \psi_0 = \sqrt{n}\, e^{i\theta}$ ← any phase.

* "Grand cannonical" ensemble (N is not fixed)

$$\boxed{\Omega = E - \mu N = \int d^3x \left[\frac{\hbar^2}{2m} |\nabla\psi|^2 + (U-\mu)|\psi|^2\right]}$$

Ground state is given by ψ_0 that minimize Ω (no other constraint)

Eq. of motion: (EoM)

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + U - \mu\right)\psi}$$

How to find ψ_0 (and relation between Ω and EoM)

$$\delta\Omega = \int d^3x\ \frac{\hbar^2}{2m}\nabla\delta\psi^*\nabla\psi + (U-\mu)\,\delta\psi^*\psi$$
$$+ \frac{\hbar^2}{2m}\nabla\psi^*\nabla\delta\psi + (U-\mu)\,\psi^*\delta\psi + O(\delta\psi^2)$$
$$= \int d^3x\, \delta\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2 + U-\mu\right)\psi + \delta\psi\left(-\frac{\hbar^2}{2m}\nabla^2 + U-\mu\right)\psi^*$$

$$\Rightarrow \boxed{\left[-\frac{\hbar^2}{2m}\nabla^2 + (U-\mu)\right]\psi_0 = 0} \quad (*)$$

↑ Right hand side of equation of motion

How to use:

(a) Pick a μ. (b) Find a ψ_0 that satisfies (*)
(c) check if $\int d^3x\, |\psi_0|^2 = N$ (d) if not find another ψ_0 and/or pick a new μ.

Example: $U =$ const.

(1) $\mu < U$: $\left(-\frac{\hbar^2}{2m}\nabla^2 + \underbrace{U-\mu}_{>0}\right)\psi_0 = 0$

has only one solution $\psi_0 = 0$
$\Rightarrow$ No bosons

(2) $\mu = U$: $\left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi_0 = 0$

has many solutions $\psi_0 = c\, e^{i\theta}$
$c, \theta = \forall$ numbers.
$\Rightarrow$ any number bosons

(3) $\mu > U$ $\left[-\frac{\hbar^2}{2m}\nabla^2 + \underbrace{(U-\mu)}_{<0}\right]\psi_0 = 0$

has many solutions $\psi_0 = c\, e^{i\vec{k}\cdot\vec{x}}$

$$\boxed{\frac{\hbar^2 k^2}{2m} = \mu - U}$$

But $\Omega = \int d^3x\, \psi_0^* \left(-\frac{\hbar^2}{2m}\nabla^2 + U - \mu\right)\psi_0 = 0$

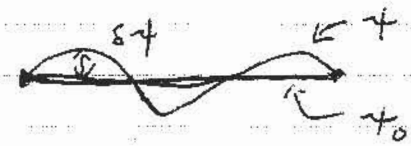
if $\psi_0 = \sqrt{n} \Rightarrow \Omega = \int d^3x\; n\,(U-\mu) < 0$
Thus $\psi_0 = c\, e^{i\vec{k}\cdot\vec{x}}$ is a maximum NOT minimum

To get a N-boson state we must set $\mu = U$ and <u>choose</u> $\psi_0 = \sqrt{n}\, e^{i\theta}$

$n = \frac{N}{V}$

(In general there is only one solution ψ_0 for each choice of μ. We need to tune μ to make $\int d^3x\, |\psi_0|^2 = N$)

☆ Collective excitations



$$\psi = \psi_0 + \underbrace{\delta\psi}_{\text{excitations}}$$

$\psi_0(x,t)$, satisfies $i\hbar \frac{\partial \psi_0}{\partial t} = \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + U - \mu\right)\psi_0}_{=0}$

$$\Rightarrow \psi_0 = \psi_0(\vec{x})$$

no t dependence

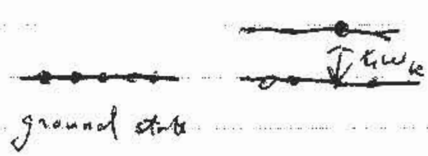
$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + U - \mu\right)\psi$$

$$\Rightarrow \boxed{i\hbar \frac{\partial \delta\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + U - \mu\right)\delta\psi}$$

Eq. of motion for excitations

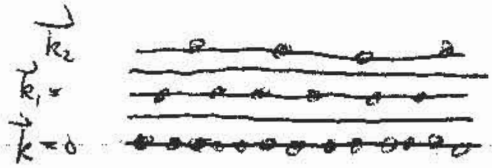
example $U = $ const. $\mu = U$

$$\Rightarrow \delta\psi = * \, e^{i\vec{k}\cdot\vec{x} - i\omega_k t}$$



$\hbar\omega_k = \frac{\hbar^2 k^2}{2m}$ ← energy of excitation (one excited boson)

$\hbar\vec{k} =$ momentum of excitation



More general:

$$\psi(x,t) = \underset{\uparrow\, c_0 e^{i\theta}}{\psi_0} + c_1 e^{i\vec{k}_1\cdot x - i\omega_1 t} + c_2 e^{i\vec{k}_2\cdot\vec{x} - i\omega_2 t}$$

$|c_0|^2$ density of bosons in the $\vec{k}=0$ level

$|c_1|^2$ " " " " $\vec{k}=\vec{k}_1$ level

$\left.\begin{array}{ll} \hbar\vec{k}_1 & \text{momentum} \\ \hbar\omega_1 & \text{energy} \end{array}\right\}$ of one boson

★ Interacting bosons.

$$\Omega = \int d^3x \left[\frac{\hbar^2}{2m} |\nabla \psi|^2 + U |\psi|^2 \right] + \int d^3x\, d^3x' \; \frac{1}{2} V(\vec{x}, \vec{x}') \underbrace{|\psi(\vec{x})|^2}_{\rho(\vec{x})} \underbrace{|\psi(\vec{x}')|^2}_{\rho(\vec{x}')} - \int d^3x\, \mu |\psi|^2$$

Eq. of motion

$$i\hbar \frac{\partial}{\partial t} \psi = \left(-\frac{\hbar^2}{2m} \nabla^2 + U_{eff} - \mu \right) \psi$$

non-linear S-eq.

$V(\vec{x}, \vec{x}')$ potential between a boson at $\vec{x}$ and a boson at $\vec{x}'$

$$U_{eff}(\vec{x}, \psi) = U(\vec{x}) + \int d^3\vec{x}' \; V(\vec{x}, \vec{x}') \, |\psi(\vec{x}')|^2$$

Ground state ψ_0 minimize Ω : $\left[-\frac{\hbar^2}{2m} \nabla^2 + U_{eff}(x, \psi_0) - \mu \right] \psi_0 = 0$

★ Short range interaction $\quad V(\vec{x}, \vec{x}') = U_0 \, \delta^3(\vec{x} - \vec{x}')$

$$\Omega = \int d^3x \left[\frac{\hbar^2}{2m} |\nabla \psi|^2 + (U - \mu) |\psi|^2 + \frac{U_0}{2} |\psi|^4 \right]$$

$$i\hbar \frac{\partial}{\partial t} \psi = \left(-\frac{\hbar^2}{2m} \nabla^2 + U - \mu + U_0 |\psi|^2 \right) \psi$$

Gross–Pitaevsky Eq.

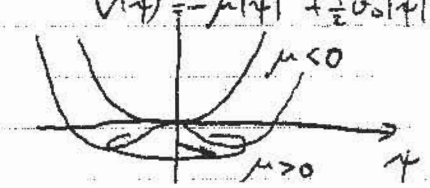
Eq. motion for collective excitations $\delta\psi = \psi - \underbrace{\psi_0(\vec{x})}_{\text{no t dep.}}$

$$i\hbar \frac{\partial}{\partial t} \delta\psi = \left(-\frac{\hbar^2}{2m} \nabla^2 + U - \mu + 2U_0 |\psi_0|^2 \right) \delta\psi + U_0 |\psi|^2 \delta\psi^* + O(\delta\psi^2)$$

★ Example: $V=0$ $\quad \Omega = \int d^3x \left(\frac{\hbar^2}{2m}|\nabla\psi|^2 \overbrace{- \mu|\psi|^2 + \frac{1}{2}v_0|\psi|^4}^{V(\psi)}\right)$

(A) Ground state: $\quad \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu + v_0|\psi_0|^2\right)\psi_0 = 0$

$\psi_0 = \underbrace{\sqrt{n}\, e^{i\theta}}_{\text{const}}$ minimize Ω

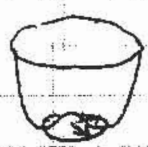


$\Rightarrow \left(-\mu + v_0 n\right)\sqrt{n}\, e^{i\theta} = 0$

$\Rightarrow \quad n = \frac{\mu}{v_0} \quad$ or $\quad 0$

① $\mu < 0$ $\quad \psi_0 = 0$, $\quad n = \frac{\mu}{v_0} < 0$ | not allowed

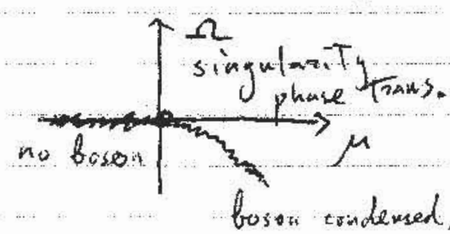
② $\mu > 0$ $\quad \underbrace{\psi_0 = \sqrt{\frac{\mu}{v_0}}\, e^{i\theta}}_{\text{symmetry breaking}}$, $\quad n = 0$ | maximum



For N-boson system $\quad \frac{N}{V} = \frac{\mu}{v_0}$ or $\boxed{\mu = v_0 n}$

$$\Omega = \begin{cases} 0 & \mu < 0 \quad \psi_0 = 0 \\ -\frac{1}{2}\frac{\mu^2}{v_0} & \mu > 0 \quad \psi_0 \neq 0 \end{cases}$$

order parameter.



$T=0$ Transition

quantum phase trans.

(B) Collective excitations

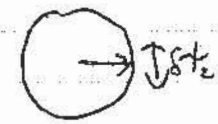
$$i\hbar\frac{\partial}{\partial t}\delta\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \mu\right)\delta\psi + \mu\,\delta\psi^*$$

$$\delta\psi = \delta\psi_1 + i\,\delta\psi_2$$

$$\begin{cases} -\hbar\frac{\partial}{\partial t}\delta\psi_2 = \left(-\frac{\hbar^2}{2m}\nabla^2 + 2\mu\right)\delta\psi_1 \\ \hbar\frac{\partial}{\partial t}\delta\psi_1 = -\frac{\hbar^2}{2m}\nabla^2\,\delta\psi_2 \end{cases}$$

$$-\hbar^2\frac{\partial^2}{\partial t^2}\delta\psi_2 = \left(-\frac{\hbar^2}{2m}\nabla^2 + 2\mu\right)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\delta\psi_2$$

$$\begin{cases} \delta\varphi_2 = c \quad \sin(\vec{k}\cdot\vec{x} - \omega_k t + \theta) \\ \delta\varphi_1 = \dfrac{\hbar k^2}{2m\,\omega_k} \; c \; \cos(\vec{k}\cdot\vec{x} - \omega_k t + \theta) \end{cases}$$



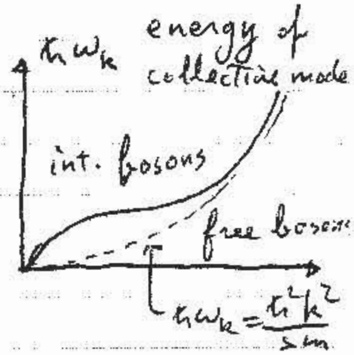
$$\boxed{\omega_k^2 = \left(\frac{\hbar^2}{2m} k^2 + 2\mu\right)\frac{k^2}{2m}}$$

$$\omega_{\vec{k}} = \sqrt{\frac{\vec{k}^2}{2m}\left(\frac{\hbar^2}{2m} k^2 + 2\mu\right)} = \sqrt{\frac{\vec{k}^2}{2m}\left(\frac{\hbar^2}{2m}\vec{k}^2 + 2v_0 n\right)}$$

$$\underset{\text{small } \vec{k}}{\simeq} v_s|\vec{k}| \qquad \boxed{v_s = \sqrt{\frac{\mu}{m}} = \sqrt{\frac{v_0 n}{m}}}$$

→ sound velosity

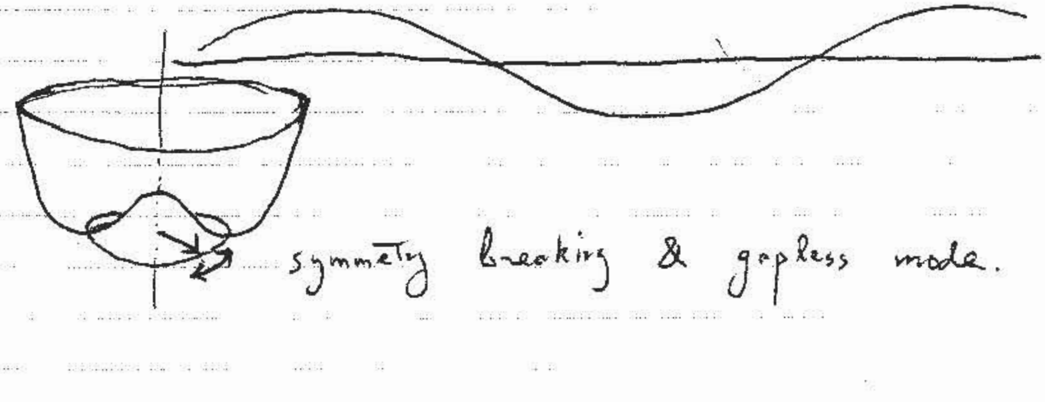
$$\underset{\text{large } \vec{k}}{\simeq} \frac{\hbar \vec{k}^2}{2m}$$



Low energy excitations $\hbar\omega_k = v|\vec{k}|$

not $\epsilon_k = \frac{\hbar^2 k^2}{2m}\Big|_{\text{free boson}}$

$C_V \propto T^3$ at low T not $C_V \propto T^{3/2}\Big|_{\text{free bosons}}$



symmetry breaking & gapless mode.