8.07 Lecture 11
October 1, 2012

Separation of Variables in Spherical Coordinates:

Laplacian:

$$\nabla^2 \varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \varphi}{\partial \phi^2}$$

Try writing $\quad \varphi = R(r)\, F(\theta,\phi)$

Then

$$0 = \frac{r^2}{RF}\nabla^2 \varphi = \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{-C_\theta} + \underbrace{\frac{1}{F}\nabla_\theta^2 F}_{C_\theta}$$

where

$$\nabla_\theta^2 F \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 F}{\partial \phi^2}$$

∇_θ^2 could be called $\nabla_{\theta,\phi}^2$, but that's too hard to write.

Want to find $F(\theta, \phi)$ such that

$$\nabla_\theta^2 F = C_\theta F$$

Treatment to be used here:

1) More general than Griffiths: he considers only solutions independent of ϕ — azimuthal symmetry.
2) Different from Jackson and other textbooks. I will be describing spherical harmonics in terms of traceless symmetric tensors. I think this makes it easier to understand what the spherical harmonics are.

Claim: The most general function of angles θ, ϕ can be written as a power series in $\hat{n}$ (unit vector in direction of (θ, ϕ)):

$$F(\hat{n}) = C^{(0)} + C^{(1)}_i \hat{n}_i + C^{(2)}_{ij} \hat{n}_i \hat{n}_j + \dots$$

where repeated indices are summed 1 to 3 (as Cartesian coordinates) and

$$\hat{n}_1 = \hat{n}_x = \sin\theta \cos\phi$$
$$\hat{n}_2 = \hat{n}_y = \sin\theta \sin\phi$$
$$\hat{n}_3 = \hat{n}_z = \cos\theta$$

Prelude: the term $C^{\ell}_{i_1 \dots i_\ell} \hat{n}_{i_1} \dots \hat{n}_{i_\ell}$ will become the spherical harmonics $Y_{\ell m}(\theta, \phi)$, where ℓ has same meaning in both expressions.

Can impose restrictions on C's without loss of generality:

1) Can always choose $C^{(2)}_{ij}$ symmetric. Why: because an antisymmetric part of $C^{(2)}_{ij}$ would not contribute to $F(\hat{n})$

2) Can always choose $C^{(2)}_{ij}$ to be traceless — $C_{ii} = 0$

Why:

Suppose $C_{ii} = \lambda \neq 0$.

Then we can define $\tilde{C}_{ij} \equiv C_{ij} - \frac{1}{3}\lambda\,\delta_{ij}$

Then $\tilde{C}_{ii} = C_{ii} - \frac{1}{3}\lambda\,\delta_{ii}$

$$= \lambda - \tfrac{1}{3}\lambda(3) = 0$$

and

$$C_{ij}\,\hat{n}_i\hat{n}_j = \left[\tilde{C}_{ij} + \tfrac{1}{3}\lambda\delta_{ij}\right]\hat{n}_i\hat{n}_j$$

$$= \tilde{C}_{ij}\,\hat{n}_i\hat{n}_j + \underbrace{\tfrac{1}{3}\lambda}_{\text{Can add to } C^{(0)}}.$$

I used $C^{(2)}_{ij}$ to illustrate, but for all ℓ, $C^{(\ell)}_{i_1 \ldots i_\ell}$ can be chosen traceless ($C^{(\ell)}{}_{i_1 \ldots i_{\ell-2}\, jj} = 0$) and symmetric. (These conditions are empty for $C^{(0)}$ and $C^{(1)}_i$, but we use a vocabulary in which $C^{(0)}$ and $C^{(1)}_i$ are called traceless and symmetric.)

Evaluation of $\nabla_\theta^2 F(\hat{n})$.

Trick: introduce radial variable r, with

$$r\hat{n} \equiv \vec{r} = x_i \hat{e}_i$$

unit vectors

$$= x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$$

Given any $F(\hat{n})$, define

$$F(r\hat{n}) = F(\vec{r}) = C^{(0)} + C_i^{(1)} x_i + C_{ij}^{(2)} x_i x_j + \dots + C^{(\ell)}_{i_1 i_2 \dots i_\ell} x_{i_1} x_{i_2} \dots x_{i_\ell} + \dots$$

We'll calculate $\nabla^2 F(\vec{r})$ first, and then infer $\nabla_\theta^2 F(\hat{n})$.

Calculate term by term:

$$\nabla^2 C^{(0)} = 0 \quad \text{(of course)}$$

$$\nabla^2 C_i^{(1)} x_i = 0$$

(1st derivate gives a constant, so 2nd derivative vanishes)

$$\nabla^2 C^{(2)}_{ij} x_i x_j = C^{(2)}_{ij} \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^k} (x_i x_j)$$

$$= C^{(2)}_{ij} \frac{\partial}{\partial x^k} \left[\delta_{ik} x_j + \delta_{jk} x_i \right]$$

$$= 2 C^{(2)}_{ij} \left[\delta_{ik} \delta_{jk} \right] = 2 C^{(2)}_{ij} \delta_{ij} = 0$$

by traceless condition.

$$\nabla^2 C^{(3)}_{ijk} x_i x_j x_k = C^{(3)}_{ijk} \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^m} (x_i x_j x_k)$$

$$= C^{(3)}_{ijk} \frac{\partial}{\partial x^m} \left(\delta_{im} x_j x_k + \text{similar terms} \right)$$

$$= C^{(3)}_{ijk} \left(\delta_{im} \delta_{jm} x_k + \text{similar terms} \right)$$

$$= C^{(3)}_{ijk} \left(\delta_{ij} x_k + \text{similar terms} \right)$$

$$= 0$$ by tracelessness.

Works in general: $\nabla^2 F(\vec{r}) = 0$.

But

$$\nabla^2 F(\vec{r}) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2} \nabla_\theta^2 F(\vec{r})$$

Look at ℓ'th term of F:

$$F_\ell(\vec{r}) = C^{(\ell)}_{i_1 \ldots i_\ell}\, x_{i_1} \cdots x_{i_\ell}$$

$$= r^\ell\, C^{(\ell)}_{i_1 \ldots i_\ell}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$$

$$= r^\ell F_\ell(\hat{n})$$

So

$$\nabla^2 F_\ell(\vec{r}) = 0 = \frac{1}{r^2}\nabla^2_\theta F_\ell(\vec{r}) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial F_\ell(\vec{r})}{\partial r}\right)$$

$$= r^{\ell-2}\,\nabla^2_\theta F_\ell(\hat{n}) + \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d r^\ell}{dr}\right) F_\ell(\hat{n})$$

But

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dr^\ell}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(\ell r^2 r^{\ell-1}\right)$$

$$= \ell(\ell+1)\, r^{\ell-2}$$

So $\nabla^2 F_\ell(\vec{r}) = 0 \implies$

$$\boxed{\nabla^2_\theta F_\ell(\hat{n}) = -\ell(\ell+1) F_\ell(\hat{n})}$$

We have found the eigenfunctions ($F_\ell(\hat{n}) = C^{(\ell)}_{i_1 \ldots i_\ell}\, \hat{n}_{i_1 \ldots i_\ell}$) and eigenvalues ($-\ell(\ell+1)$) of ∇^2_θ!