8.07 Lecture 6
Monday, September 17, 2012

Our goal is to discuss the integral formulation of the following differential equations of electrostatics,

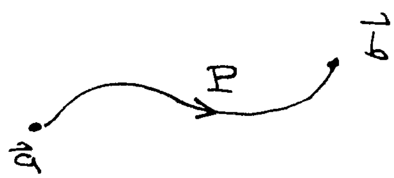
$$\vec{E} = -\vec{\nabla} V :$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{E} = 0,$$

and some related consequences.

Integral form of $\vec{E} = -\vec{\nabla} V$:



$$\underset{P}{\int_{\vec{a}}^{\vec{b}}} \vec{E}\cdot d\vec{\ell} = -\int_{\vec{a}}^{\vec{b}} \vec{\nabla} V\cdot d\vec{\ell}$$

$$= V(\vec{a}) - V(\vec{b})$$

This relation provides an alternative definition of V:

$$\boxed{V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot d\vec{\ell}\,'}$$

$\vec{r}_0$ = arbitrary reference point

By this definition, $V(\vec{r})$ has an arbitrary 0-point. Adding a constant to $V(\vec{r})$ has no significance. Only potential differences are meaningful.

The previous definition for $V(\vec{r})$ as an integral over $\rho(\vec{r})$ gives

$$\lim_{|\vec{r}|\to\infty} V(\vec{r}) = 0$$

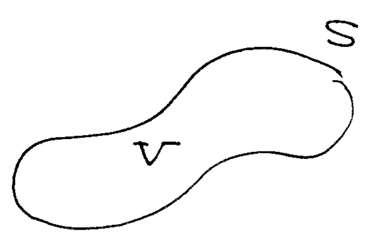
so it corresponds to using $\vec{r}_0 = \infty$, $V(\vec{r}_0) = 0$.

For idealized textbook problems (e.g. infinite line of charge) the expression for $V(\vec{r})$ as an integral over $\rho(\vec{r})$ may not converge. The line integral formula above, however, can still be used to define potential differences.

Integral form of Gauss's law:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$$

Integrate over volume:



$$\int_V \vec{\nabla}\cdot\vec{E}\, d^3x = \oint_S \vec{E}\cdot d\vec{a}$$

$$\therefore \oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int_V d^3x\,\rho = \frac{1}{\epsilon_0}\int dq$$

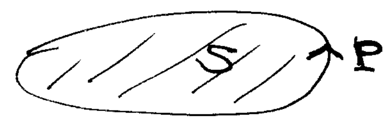
$$\boxed{\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enclosed}}$$

True even for dynamics.

Integral form of $\vec{\nabla}\times\vec{E} = 0$:

$$\vec{\nabla}\times\vec{E} = 0$$

Integrate over surface



$$\int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a} = \oint_P \vec{E}\cdot d\vec{l}$$

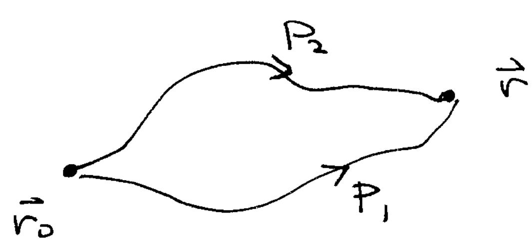
So $\boxed{\oint \vec{E}\cdot d\vec{l} = 0}$

Ambiguity in potential formula.

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot d\vec{l}\,'$$

What path?

Ans: it doesn't matter.



$$\int_{P_1} \vec{E}\cdot d\vec{l} - \int_{P_2} \vec{E}\cdot d\vec{l} = \oint \vec{E}\cdot d\vec{l} = 0$$

Work and Energy.

Work done by $\vec{E}$ on charge q moving from $\vec{a}$ to $\vec{b}$:

$$W_{\vec{E}} = \int \vec{F} \cdot \vec{d\ell} = q \int \vec{E} \cdot \vec{d\ell}$$

$$= -q \left[V(\vec{b}) - V(\vec{a}) \right]$$ independent of path

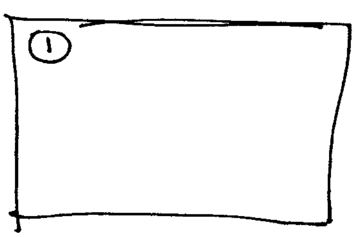
Work that I must do to move particle from $\vec{a}$ to $\vec{b}$ (opposing $\vec{E}$)

$$W_{me} = -W_E = q \left[V(\vec{b}) - V(\vec{a}) \right].$$

So $qV(\vec{r})$ = potential energy of charge q at $\vec{r}$

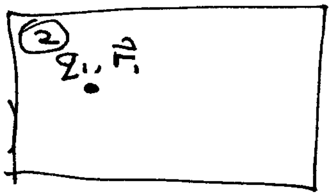
for moving 1 charge in fixed background of other charges.

Building up a configuration of point charges:

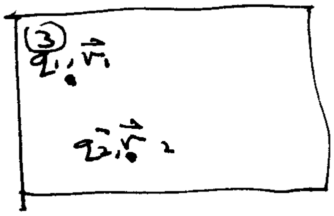


Nothing

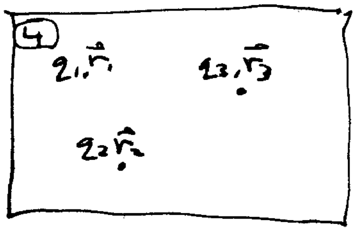
$W = 0$ (work done by me)



$W = 0$ (don't include work to make charge q_1)



$$W = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$



$$\Delta W = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

$$W_{tot} = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

For n-charges:

$$W_{tot} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}\frac{q_i q_j}{r_{ij}}$$

or

$$\boxed{W_{tot} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{i,j=1 \\ i\neq j}}^{n}\frac{q_i q_j}{r_{ij}}}$$

For continuous charge density $\rho(\vec{r})$.

$$q_i \rightarrow \rho(\vec{r})\, d^3x$$

$$\boxed{W = \frac{1}{8\pi\epsilon_0}\int d^3x\, d^3x' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}}$$

If $\rho(\vec{r})$ is a smooth function, then excluding $\vec{r}\,' = \vec{r}$ makes no difference to the integral.
If $\rho(\vec{r})$ includes δ-functions, then W includes the self-energy of these point-charges, and diverges. Then it is easier to use the discrete sum excluding $i=j$.

Further ways of writing W:
Re-ordering,

$$W = \frac{1}{2}\int d^3x\, \rho(\vec{r}) \underbrace{\frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}_{V(\vec{r})}$$

so

$$\boxed{W = \frac{1}{2}\int d^3x\, \rho(\vec{r})\, V(\vec{r})}$$

Another way to write W:

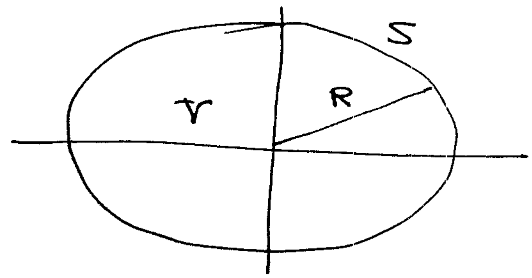
Use $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0}$, so

$$W = \frac{1}{2}\epsilon_0 \int d^3x\; V\, \vec{\nabla}\cdot\vec{E}$$

$$= \frac{1}{2}\epsilon_0 \int d^3x \left[\vec{\nabla}\cdot(V\vec{E}) - \vec{\nabla}V\cdot\vec{E}\right]$$

$$= \frac{1}{2}\epsilon_0 \int_{\underset{\text{volume}}{V}} d^3x\, |\vec{E}|^2 + \frac{1}{2}\epsilon_0 \int_{\underset{\text{surface}}{S}} V\vec{E}\cdot d\vec{a}$$

Take V as a sphere of radius R, centered at the origin:



Estimate how the factors in the surface term behave as $R \to \infty$.

$$\int_S V \vec{E} \cdot d\vec{a} \xrightarrow[R\to\infty]{} 0$$

$V \sim \frac{1}{R}$, $\quad \vec{E} \sim \frac{1}{R^2}$, $\quad \int |d\vec{a}| \sim R^2$

Thus, the surface integral vanishes as $R \to \infty$, at least for any finite charge configuration, so

$$\boxed{W = \frac{1}{2}\epsilon_0 \int_{\substack{\text{all} \\ \text{space}}} |\vec{E}|^2 \, d^3x}$$

Newtonian gravity analogy:

Newton's law of gravity looks like Coulomb's law, except for the constants.

But the sign is different! Positive masses attract each other.

Result: the energy density of a Newtonian gravitational field is <u>negative</u>.

In GR, gravitational energy is harder to define, but to the extent that it can be defined, it is also negative.

Consequences:

1) Stars: when stars collapse, they produce a stronger gravitational field inside the volume of the star before it collapsed (no change far away). Because energy is released by creating a gravitational field, the kinetic energy increases and the star heats up.

2) Cosmology:

The gravitational field that fills the universe contributes negatively to the total energy. Despite the huge Mc^2 energy of the matter in the universe, the total energy of the universe is consistent with zero. This fact makes it possible for the inflationary universe model to build an arbitrarily large universe starting from something incredibly small.

Comment on self-energy:

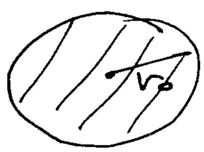
For single charge, $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$

$$W = \frac{1}{2}\epsilon_0 \frac{q^2}{(4\pi\epsilon_0)^2} \int_0^\infty (4\pi r^2 \, dr) \frac{1}{r^4}$$

$$= \infty$$

Status of infinite self-energy:

Classically, can build models that work for some questions, but not all:



$$m_{phys} = m_0 + \frac{\frac{1}{2}\epsilon_0 \int |\vec{E}|^2 d^3x}{c^2}$$

Take limit as $r_0 \to 0$, m_{phys} fixed ($m_0 \to -\infty$!).

Gives the Abraham-Lorentz formula for the radiation-reaction force:

$$\vec{F}_{rad} = \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}$$

For an oscillating charge, $\vec{v}$ and $\dot{\vec{a}}$ point in opposite directions, so the formula describes damping, which correctly compensates for the radiated energy.

But: if I push on a charged particle to start it accelerating, $\vec{F}_{rad}$ points in same direction as $\vec{v}$, and there are runaway solutions. These runaway solutions are clearly an error associated with treating the point charge.

In QED (quantum electrodynamics), the problem of point-charge infinities is under better control. Will discuss next time.