PROFESSOR: We're finished with WKB. In recitation, you saw some transmission across the barrier and that's also included in the notes. That's an important application of WKB and should look at it.

And today we're going to start with a new topic. It's time-dependent perturbation theory. And time-dependent perturbation theory is going to keep us busy for a number of lectures. There's a lot of applications of these ideas and the methods are rather general. Here is the place where we will learn about Fermi's Golden Rule. The kind of rule that is useful for radiation problems, ionization problems, transitions. It's all very applied. Nevertheless, we have to develop the theory carefully and see what are the main concepts.

So time-dependent perturbation theory is our subject. Time-dependent perturbation theory. Again, we will begin with a Hamiltonian that is time independent. And we know about it. We call this Hamiltonian H zero. The same name we used for time-independent perturbation theory. We have H zero.

This time, however, we will have a perturbation. The perturbation will also be called delta H. But the big difference is this time the perturbation will be time dependent. And this will be our whole Hamiltonian. So this is the subject we're trying to understand.

Whenever you have a Hamiltonian, this time independent, we already know how we're supposed to deal with it. We're supposed to find the energy eigenstates. And then we'll have the whole collection of energy eigenstates. We can solve any problem. In the initial condition of the wave function, you expand it in energy eigenstates. You evolve the state in time. Everything is nice and simple.

In a sense, for the first time in your studies of quantum mechanics at MIT, we're going to face very directly the difficulties of time dependence. And the first difficulty of a time dependent Hamiltonian is that you cannot define energy eigenstates anymore. The whole concept is gone. That's pretty radical. But it's unfortunately the truth.

When you first learn in 804, how to work with energy eigenstates, it was all dependent on a factorization. A possible factorization of the solution in the factor that depends on position and a function that depends on time. And that time-dependence was always very simple. E to the minus iEt over h-bar, where E was the energy which was a solution of the time-independent, spatial part of the problem.

So separating the differential equation was possible because, for example, the potentials never depended on time. If the potential depends on time, imagine the equation H psi equal e psi. This supposed to be time independent. But H has a time dependent potential. It's just impossible.

So we don't have energy eigenstates, any more, for this potential, for the new potential, for the new Hamiltonian. And we'll have to think how we're going to face these difficulties. So no energy eigenstates for H of t. Of course, there are energy eigenstates for H0. H0 is supposed to be your time-independent Hamiltonian.

So how do we think of this? We will think of time, evolving here, and typically we will have an initial time t0. And throughout this region, the Hamiltonian, H is equal to H0, for time less than 10 0. Then we imagine that the perturbation turns on, and suddenly things start happening, up to time tf. Here the Hamiltonian is H0 plus delta H t. And after, the time tf, the Hamiltonian, is back to H0.

So that's a nice way of thinking about the problem, in which we imagine, OK, perturbation is localized that some time t begins there before you have the known Hamiltonian, after you have the known Hamiltonian. That allows you to rephrase questions in a clearer way, because there are energy eigenstates here.

If you sit at that energy eigenstate before time equal t0, you will remain in an energy eigenstate. There's no reason why it changes. That's what time-independent Hamiltonians do. The energy eigenstate changes by a phase. And that's all it does. So here, we can speak about energy eigenstates, and after time tf, we can speak about energy eigenstates.

So we can ask the question, suppose you were sitting on this particular energy eigenstate, here. Then the world shakes for a few moments. Which state are you going to find yourself, after this process? And this is a question of going from one energy aide and state to some collection of energy eigenstates. By the time the lecture will be finished, we will have solved this problem in principle, and set up how you would do it, in practice, for any case.

This is a very physical way of thinking, as well. You can imagine, you have a hydrogen atom in its ground state. So the original system is a Hamiltonian for a particle, an electron, and a proton. And it happens to be in a ground state. Then you send in an electromagnetic wave, something we will do next lecture. No, in a couple of lectures.

And then, it's possible that the atom gets ionized, or that the electron is kicked up to a higher level. You will be able to calculate those transition functions. You will be able to calculate the probability of ionization. In which, after the perturbation is all said and done, you ask, what is the probability that you're in an eigenstate, that this a higher excited state of the hydrogen atom?

And with a little bit of flexibility in your mind, you can think of the hydrogen atom as a system that has bound states, and continuum states, in which you have a proton and an electron traveling a wave. If you your electron is far enough from the proton, it's like a plane wave. If it's not that far, it can travel. And its wave is deformed a little. Those are the continuum states of the hydrogen atom.

So you could ask, what is the probability that it's ionized, and it's a transition to a continuum eigenstate in the hydrogen atom? This sounds more complicated. It's a little more complicated. Why? Because when you go from one state to another, you can count it.

But when you ask what is the probability that the electron goes into the continuum of plane wave, you'll have infinitely many continuum states. And we will have to deal with that. Where we will know-- we will not be able to avoid this complication. And that complication makes for a very interesting result, transitions to the continuum.

So this is what we want to do. And if given, that in general, this is not a problem of finding energy eigenstates, the energy eigenstates are known, we want to find the wave function, psi of t. That's our real unknown. And to do that, we will use something called the interaction picture.

Just like we have Heisenberg picture, Schrodinger picture, we'll have an interaction picture. Don't worry. It's not more complicated than anything you've seen. It's, in fact, a very sensible way of doing things, in which you combine good things from the Heisenberg picture, and nice things from the Schrodinger picture, together.

So remember, a little of what was going on with the Heisenberg picture. It all began by saying that, if you have the expectation value of a Schrodinger operator, that was the same thing. A s for A Schrodinger, that was possible to compute as the expectation value of the Heisenberg operator on the time equals 0 states, in the states that don't vary.

You don't need to compute expectation values of operators used in the time involved states.

You can think of time involved operators and compute in this expectation value in the time equals 0 states. For that, you use the unitary operator, that does time evolution, u of t psi at 0 u dagger of t psi at 0.

And this unitary operator, it's in general difficult to calculate. It's a unitary operator that does time evolution and, in our case, it's hard because the Hamiltonian is complicated. But in general, we call this the Heisenberg operator. A Heisenberg of t. That's the definition of the Heisenberg operator.

Another concept that is kind of useful, is the idea of operators that brings states to rest. So suppose you have the state psi at sum time t. I want to act with an operator that will bring it to rest. That means that this is time-dependent. I want to act with something that will make it time-independent. So the answer is U dagger. U dagger is a unitary operator, so it's the inverse of U.

So look at this expression, U dagger on this state gives you U dagger. The state is U on the state at time equals 0. U dagger U, being unitary, it's just the unit matrix, and your 2 0. So this operator, U dagger, removes the time-dependence. The [INAUDIBLE] uses the expression, brings the operator to rest.

So we can think of doing something of this kind. You see, the whole Hamiltonian is H0 plus delta H. H0 you understand well. Delta H is complicated. So how about doing the time evolution through Heisenberg, with H0, with what you know. You don't know the full U, but you know the U that would do the time evolution, for the time-independent Hamiltonian, H0.

So let's attempt to do the part that is easy. You see, there's going to be time evolution as you go from t0 to tf. Some of that is going to be generated by H0, some of that by delta H. Let's let Heisenberg do the work for H0, and Schrodinger do the work for delta H. That's basically the idea. .

You see, you know that what is difficult about this problem is the delta H. So you solve the Schrodinger equation. Let's solve the Schrodinger equation that just has delta H, doesn't have H anymore. So motivated by this, we'll do exactly that. Think of for H0, the operator U is e to the minus i H0 t over h-bar. This is the operator that generates time evolution for H0.

So we will take the state psi of t and remove the time-dependence associated to H0. So try to bring the state to rest. So we'll put here, e to the minus-- I'm sorry. I'm supposed to put U

dagger. So I'll put e to the I H0 t over H-bar. And look at that. This is U dagger. That's the kind of thing that brings the state to rest.

If the Hamiltonian had only been H0, only H0, this would be time-independent. H0 brings it to rest. Because the Hamilton is not just H0, this will not be, in general, time-independent. But it will depend on time. So this is a kind of a nice wave function in which you sort of have removed the time evolution having to do with H0.

So we will define this as our auxiliary variable, psi of t. That's the definition. It's motivated by the idea that, if delta H was equal to 0, then psi tilde is constant in time.

Because if delta H was equal to 0, all the evolution is created by H0. You would put here, oh, this is e to the minus i H0 times psi of t equal 0, the exponentials would cancel, and everything would be simple. So this is a wave function that is going to be our new variable. We wanted to find psi of t.

Now you can say your task is find psi tilde of t. That's your new task. And it's an equally good task, because if you find psi tilde of t, then you can write psi of t as, from this equation, e to the minus i H0 t over H-bar psi tilde of t. So our task now will be to find psi tilde. If we have psi tilde, we have psi, as well. So we haven't lost any information. And this is all good.

So let's try to see what equation is satisfied by psi tilde, what kind of Schrodinger equation is satisfied by it. So what is the Schrodinger equation for psi tilde. I'll just take i H-bar d dt of psi tilde and see what I get. OK. If I have to differentiate this term, i H-bar d dt. I have to differentiate this exponential.

And the i is going to give you a minus sign. The H's are going to cancel. And this is just going to bring an H0 down. So I'm going to get minus H0 times that exponential. And that exponential times psi of t is psi tilde of t. So the derivative of the first term gives me something with H0. And the face times that is still psi tilde.

Now I have to differentiate the second one so I have plus e to the i H0 t over H-bar. And i H-bar d dt of this cat. But that's the Schrodinger equation for the original cat. So I should put here the full Hamiltonian, H0 plus delta H times psi of t.

So when the i dd H hits the state, you get the full Hamiltonian time psi of t. But I actually were right for psi of t e to the minus i H0 t over H-bar psi tilde of t. Because I'm looking for an

equation for psi tilde. I'm going a little-- I'm speaking slowly, but going a little fast.

Now what happens? What you wanted to happen, happened. H0 is here. And look, H0 is here. Well it's accompanied by these two exponentials, but they have H0. So they commute through and these two exponentials cancel as far as this first term is concerned. So nice consolation over here.

And then we get the following equation, i H-bar d dt of psi tilde is equal to, well, this delta H sandwiched in these two operators. So I'll just copy it. e to the H0 t over H-bar delta H e to the minus i H0 t over H-S bar times pi tilde of t.

So this equation makes what we wanted manifest. If delta H vanishes, psi tilde is time-independent. But it's more than that. This which we will call delta H tilde. So tilde objects are objects that have been acted by H0, like the tilde state, it has an H0 with respect to the other. This delta H tilde is because it has been acted by similarity with those things.

But even more, I think you should realize that this thing is really the Heisenberg 0 version of delta H. A Heisenberg operator is obtained by taking the Schrodinger operator, putting U dagger and U. And that's exactly what you've done here.

You've taken the Schrodinger operator and put U dagger with respect to H0, and U for H0. So this is the Heisenberg version of delta H relative to H0. So delta H has been "Heisenberg-ed" using H0. And then this whole thing looks like i H-bar d dt of psi tilde equal delta H tilde psi tilde. And this is a Schrodinger equation.

So there it is, for you, the so-called interaction picture. The interaction picture says whatever is not an interaction will make it Heisenberg. Whatever is purely interactive will make it Schrodinger. And therefore, this state varies in time. And there are some operators that have acquired extra time-dependence, as well, due to the Heisenberg process.

So this is the situation we are going to try to solve.