PROFESSOR: We're going to complete our study of WKB today. There's a lot of important things we haven't said yet. And in particular, we have not explained how it all works, really, in the sense of the connection formulas. That's the most non-trivial part of the WKB story.

And in some ways it's some of the most interesting things that you have to learn from this. It's an opportunity to learn something about differential equation, Fourier transforms, and the limits of the quantities you can really compute.

So we will begin our discussion with some reconsideration of these Airy functions. You've started to hear about, probably some in the homework, some in recitation. So this relates to this person, George Biddle Airy from the Airy functions, who lived in England for 90 years. He was the seventh Royal Astronomer. And he was an astronomer from 1835 to 1881.

Was located, his work area, in Greenwich. It's a little bit outside of London. And that's where astronomers were working hard to establish all the data associated with the Prime Meridian. France also tried to make the their own meridian the most important one. But at the end of the day, it all depended of which astronomers did the most work in measuring precise, accurate determination of times, motion of the moon, Jupiter's satellites, all kinds of things that were pretty important at that time, especially also the manufacturing of good telescopes and good clocks.

So Mr. Airy also worked in optics. And that's where he discovered these functions we're studying. And they're very relevant to the WKB approximation.

So let me remind you a little of what you've been considering in the homework and extend some of those ideas as well. So you considered a differential equation, d 2nd PSI du squared equal u PSI. Pretty innocent looking differential equation. I'm always surprised how a differential equation that looks so simple can be so intricate.

But anyway, it is. And that's the Airy equation. And it shows up for this problem that you have, for example, an infinite wall and a potential goes up like that. The bound states are controlled by solutions of this equation at the end of the day. This is after you clean up all the units and do all the work.

But it shows in other places as well. And you probably found already that this PSI function, PSI of u, is given as a constant times an integral that came from Fourier transformations of e to the ik cubed over 3 e to the iku.

And so this is our integral. This is our solution of this equation. And the nice thing is that this integral tells you a lot about the function. It's a perfectly nice definition of the function.

You may have noticed, and I think it was discussed to some degree in recitation as well, that the Fourier transform converts this equation to first order differential equation. Because this becomes p squared times PSI. Well, you know from Schrodinger equations this part of the differential equation comes from p squared.

On the other hand, multiplication by x is the same as dvp in Fourier space. Therefore, you get the dvp of PSI equal p squared PSI equation. And that equation is a first order differential equation. First order differential equations have one solution.

So you would say, OK, I tried to do this by Fourier transform, and I'm getting just half of the solutions. And that's true. The reason is that the other solution of this differential equation doesn't quite have a Fourier transform, so it's a little more difficult to get at.

But once you think of this integral, it is kind of useful to try to be a little more general and think of it in terms of the complex plane, the complex k plane. So here is the k plane. And this integral is over k from minus infinity to infinity. So this is the standard countour of integration. We can call it the countour c1.

When you're doing this integral, and you're doing this in the complex plane you may want to consider that if you go off the real line, maybe things are even better. In particular, as you're integrating u is a fixed number, it's a fixed coordinate. We usually think of u as real, but we could think of u as complex as well.

And this integral has these factories. It's an oscillatory factor that becomes faster and faster oscillating in time. But the magnitude of the integrand is 1 at every place. It oscillates very fast, but the magnitude is 1.

Not the best situation possible. You know? When you're doing an integral, you have the feeling that eventually you get 0 because it's oscillating so fast. But it would be nicer if you had a decay.

So for that, consider that this function would be k if k, if the imaginary part of k cube, is positive, the ky. If the imaginary part of k cube is positive, you have a k cube would be i times a positive number. i items i is minus 1, so you get the suppression factor, if the imaginary part of k cube is positive.

So if you're integrating over a region where the imaginary part of k cube is positive, you will be in good shape. Now, if you think of k as a complex number, e to the i theta k, k cube would be this. An imaginary part of k cube is positive requires that this angle 3i theta k be between pi and 0.

Why? Because if you have a complex number, any complex number, its imaginary part is positive if the argument of that angle goes between 0 and pi. Moreover, if you get an imaginary part positive for some angle theta k, if m of k cube is positive for some theta k, it is also positive for theta k plus 2pi over 3. Because if you add 2 pi over 3 to theta k, you change this by 2pi. And therefore, the exponential doesn't change.

From this inequality, I'm sorry, I have an i too many here. The imaginary part of this requires that the argument of this be between 0 and pi, or theta k between pi over 3 and 0. So actually, here, I should draw 0 to pi over 3. This is at 60 degrees.

And that's the region in k space where the integral is suppressed. Now, we also said that this region is unchanged if you add 2pi over 3. So if I add 2pi over 3, that's precisely 1/3 of a turn. This is 30 degrees. It goes this area into this.

And finally, if I add, again, 2pi over 3, it goes into 30, 30, 30, and to this region. I guess the sign of nuclear material, hazardous materials or something that. It's not hazardous to our health. We'll survive this lecture.

So what's happening here actually is something quite interesting. For example, you're doing this integral here. If you started doing this integral, once you are far, far away, you start going up, the integral over here is going to give nothing, because you're very far away.

And therefore, you're in the region of absolute exponential suppression. We said this thing has a positive imaginary part. And therefore, you have a suppression. It's k cubed, so it's becoming like e to the minus radius cubed. It's killed.

So it's no problem to go up here, and you get nothing. So what it means is that actually there's no singularities in the integrand either. This integral that we've done here could have been done over this contour like that. And then you could have gone like that.

You could do it in Mathematica. The answer will be the same. Or you could have gone like that. Or you could have gone here like this, also you're suppressed over here. And you're suppressed over here.

And actually, by contouring the formation, you could instead of doing this in theory, you could have come here and gone up like that. And done that. That also would have been the same result. There's nothing, the integrand is analytic here. So this you can push it down and nothing goes wrong.

Or you could have done even this integral like that. Because the integral here is 0 and the integral here is 0, so you can bring them down.

In fact, if you did this, I think, with Mathematica, Mathematica when you try to do this integral, kind of does it. But it complains a little. It's worried that, you know, it's not getting the answer right because the integrand is still big. It's oscillating very fast, but it's big.

If you go here, I think Mathematica will complain less and will give you the same answer. So this is the power of complex analysis that allows you to do this integral in all kinds of ways.

And there is something even more important about this that I will explain in the notes, but I can convey the idea here. When you prove that this is a solution, and when you went to Fourier space, you had to do an integration by parts. And an integration by parts is always dangerous.

And if you're trying to find a true solution of your problem, you have to make sure it works. P, the operator P, is supposed to be her mission. And here what happens is that you turn this into dvp. And you have to fall back this factor.

And here is where this discussion is very relevant. Because when you fold out this factor, you have to integrate. You couldn't get contributions from the boundary. And you have to make sure those contributions are 0. Otherwise, you're not solving the differential equation really.

And the happy thing about these contours, and now that you understand these regions, is that, yes, the boundary contributions are going to be 0 from the two sides because of this suppression factor. So this analysis actually turns it into a rigorous analysis.

And the part that has to do with the boundary conditions, the fact that the operations that you're doing are illegal when you try to solve the differential equation, require-- this is the details that I will not derive now-- require precisely that e to the i k cubed over 3 e to the iku vanish at the ends of the countour gamma that you're using to solve the problem.

So if you're in the wave from here to here, it should vanish at this end and at that end. Rigorously speaking, this thing only vanishes if you're a little bit up. But this is what you really need for the equation to be a solution.