

PROFESSOR: I now have that new n of t that we wrote there. I have to write it as what it is. It's i ψ n of r of t times-- I will write it here this way-- d/dt -- the dot will be replaced by the d/dt -- ψ n of r of t . And then, of course, the γ n of t will be just the integral from 0 to t of new n of t prime dt prime. So that's the next step.

Well, if you have to differentiate a function that depends on r of t , what do you have? Let me do it for a simpler case, d/dt of f of r of t . This means d/dt of a function of r_1 of t are all the ones up to r_n of t .

And what must you do? Well, you should do df/dr_1 times dr_1/dt all the way up to the df/dr_n and dr_n/dt .

You want to find the time dependence of a function that depends on a collection of time-dependent coordinates. Well, the chain rule applies.

But this can be written in a funny language-- maybe not so funny-- as the gradient sub r vector of f dotted dr vector dt . See, the gradient, in general, is d/dx_1 d/dx_2 d/dx_3 . It's a vector operator. The gradient sub r would mean d/dr_1 d/dr_2 d/dr_3 , just the gradient in this Euclidean vector space times dr dt .

So that's what I want to use for this derivative. I have to differentiate that state. And therefore, I'll write it that way. So γ n of t is equal to i , from the top line, ψ n of r of t times gradients of r acting on the state ψ n of r of t dotted with dr dt . This is dot product.

So just to make sure you understand here, you have one ket here, and you have this gradient. So that gives you capital N components, the derivative of the ket with respect to r_1 r_2 r_3 r_4 . Then with the inner product, it gives your capital N numbers, which are the components of a vector that is being dotted with this vector.

It's all about trying to figure out that this language makes sense. If this made sense to you, this should make sense, a little more, maybe a tiny bit more confusing. But maybe you should write it all out. What do you think it is? And that might help you. Or we could do that later.

So if we have that, we can go to γ n , the geometric phase. So this is \int_0^t , the integral with respect to prime time, so new m . So it's i ψ n of r of t prime-- there's lots of vectors here, gradient r vector of ψ n of r of t prime dotted dr dt prime dt prime. That's the last dt prime.

And the good thing that happened, the thing that really makes all the difference, the thing that is responsible for that conceptual thing is just this cancellation. This cancellation means that you can think of the integral as happening just in the configuration space.

This is not really an integral over time. This is an integral in configuration space because now this integral is nothing else than the integral over the path γ . Because the path γ represents the evolution of the coordinate capital R from 0 to time t . This is nothing else than the integral over the path γ of $i \psi_n$ of r -- I don't have to write the t anymore-- $dr \psi_n$ of r -- again no t -- $\cdot dr$. And this is the geometric phase γ_n that depends on r on the path. I'll write it like that.

You see, something very important has happened here. It's a realization that time plays no role anymore. This is the concept. This is what you have to struggle to understand here.

This integral says take this path. Take a little dr dot it with this gradient of this object, which is kind of the gradient of this ket, which is a lot of kets with this thing. So it's a vector. Dot it with this and integrate. And time plays no role. You just follow the path.

So whether this thing took one minute to make the path or a billion years, the geometric phase will be exactly the same. It just depends on the path it took.

Time for some names for these things. Let's see. So a first name is that this whole object is going to be called the Berry connection. $i \psi_n$ of r gradient $r \psi_n$ of r is called the Berry connection a n vector of r . Berry connection.

OK, a few things to notice, the Berry connection is like a vector in the configuration space. It has capital N components because this is a gradient. And therefore, it produces of this ket n kets and, therefore, n numbers because of the bra. So this is a thing with capital N components.

So it's a vector in R^N . But people like the name connection. Why Connection? Because it's a little more subtle than a vector. It transforms under Gage transformation, your favorite things. And it makes it interesting because it transforms under Gage transformation. We'll see it in a second.

So it's a connection because of that. And there's one Berry connection for every eigenstate of your system. Because we fix some n , and we got the connection. And we're going to get

different connections for different n 's. So n components, one per eigenstate, and they live all over the configuration space.

You can ask, what is the value of the Berry connection at this point? And there is an answer. At every point, this connection exists.

Now, let's figure out the issue of gauge transformations here. And it's important because this subject somehow-- these formulas, I think in many ways, were known to everybody for a long time. But Berry probably clarified this issue of the time independence and emphasized that this could be interesting in some cases.

But in fact, in most cases, you could say they're not all that relevant. You can change them. So here is one thing that can happen. You have your energy eigenstates, your instantaneous eigenstates. You solve them, and you box them. You're very happy with them.

But in fact, they're far from unique. Your energy eigenstates, your instantaneous energy eigenstates can be changed. If you have an energy eigenstate ψ_n of r -- that's what it is-- well, you could decide to find another one. ψ'_n of r is going to be equal to $e^{i\theta(r)}$ times ψ_n . And these new states are energy eigenstates, instantaneous energy eigenstates that are as good as your original ψ_n because this equation also holds for the ψ'_n .

If you add with the Hamiltonian, the Hamiltonian in here just goes through this and hits here, produces the energy, and then the state is just the same. The r of t 's are parameters of the Hamiltonian. They're not operators. So there's no reason why the Hamiltonian would care about this factor. The r 's are just parameters.

Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: No, they're still normalized. I should put a phase here-- thank you very much-- minus i . Thank you. Yes, I want the states to be normalized, and I want them to be orthonormal. And all that is not changed if I put them phase.

So this is the funny thing about quantum mechanics. It's all about phases and complex numbers. But you can, to a large degree, change those phases at will. And whatever survives is some sort of very subtle effects between the phases. So here I put the i and β of r is real.

PROFESSOR:

So you can say let's compute the new Berry connection associated with this new state a_n prime of r . So I must do that operation that we have up there with the news state. So I would have $i \psi_n$ of r times e to the i beta of r . That's The bra. Then I have dr and now the ket, e to the minus i beta of r ψ_n of r . So this is, by definition, the new Berry connection associated to your new, redefined eigenstates.

Now this nabla is acting on everything to the right. Suppose it acts on the state and then the two exponentials will cancel, and then you get the old connection. So there is one term here, which is just the old a_n of r .

There's all these arrows there. There's probably five arrows at least I miss on every board. Here is a 1, 2, 3, 4 5.

OK, so this is the first one, and then you have the term for this gradient acts on this exponential. When the gradient acts on the exponential, it gives the same exponential times the gradient of the exponent. The exponentials then cancel. The gradient of the exponent would give me plus i times minus i gradient of beta. Maybe I'll put the r of the r .

And then these cancel, and you have the state with itself, which gives you 1. So that's all it is, all that the second term gives you. So here we get a_n of r plus gradient r of beta of r .

So this is the gauge transformation. And you say, wow, I can see now why this is called a connection. Because just like the vector potential under a gauge transformation, it transforms with a gradient of a function. So it really transforms as a vector potential, all in this space called the configuration space, not in real space. In the configuration space it acts like a vector potential. And that's why it's called a connection.

But let's see. We have now what happens to the connection. Let's see what happens to the Berry's phase if you do this. So the Berry's phase over there is this integral. So the Berry's phase can change. And let's see what happens to the Berry's phase.

So what is the geometric phase γ_n of γ ? In plain language, it is the integral over γ -- from here, I'm just copying the formula-- of a_n of r , the Berry connection, times dr .

So what is the new Berry phase for your new instantaneous energy eigenstates? Now you would say, if the Berry phase is something that is observable, it better not depend just on your convention to choose the instantaneous energy eigenstates.

And this is just your convention. Because if a problem is sufficiently messy, I bet you guys would all come up with different energy eigenstates because the phases are chosen in different ways. So it better not change if the Berry phase is to be significant.

So what is the prime thing? Well, we still integrate over the same path, but now the prime connection-- but that is the old connection $\mathbf{a} \cdot d\mathbf{r}$, the old Berry's phase, plus the integral over γ , or I will write it from initial to final \mathbf{r} . Maybe I should have i and f in the picture.

If you want to, you can put this \mathbf{r} of time equals 0 as \mathbf{r}_i and \mathbf{r} of time equal t_f is \mathbf{r}_f the extra term, the gradient of $\beta \cdot d\mathbf{r}$. So this is the old Berry phase. So the new Berry phase is the old Berry phase.

And how about the last integral? Does it vanish? No, it doesn't vanish. It gifts you. But in fact, it can be done. This is like derivative times this thing, so it's one of those simple integrals. The gradient times the d represents the change in the function as you move a little $d\mathbf{r}$. So when you go from \mathbf{r}_i to \mathbf{r}_f , the integral of the gradient is equal to the function β at \mathbf{r}_f minus the function β on \mathbf{r}_i .

This is like when you integrate the electric field along a line, and the electric field is the gradient of the potential. The integral of the electric field through a line is the potential here minus the potential there. So here this is plus β or \mathbf{r}_f minus β of \mathbf{r}_i .

So it's not gauge invariant in the Berry phase. And therefore, it will mean that most of the times it cannot be observed. It's not gauge invariant. Whatever is not gauge invariant cannot be observed. You cannot say you make a measurement and the answer is gauge-dependent because everybody is going to get a different answer. And whose answer is right? That's not possible. So if this Berry phase seems to have failed a very basic thing, then it's not gauge-invariant.

But there is one way in which this gets fixed. If your motion in the configuration space begins and ends in the same place, these two will cancel. And then it will be gauge-invariant. So the observable Berry's phase is a geometric phase accumulated by the system in a motion in a configuration space where it begins and ends in the same point. Otherwise, it's not observable. You can eliminate it.

And so this is an important result that the geometric Berry phase for a closed path in the

configuration space is gauge-invariant.