

**PROFESSOR:** Let's do it biogeometrically. There's a nice geometric interpretation to this thing.

So it comes from thinking of this in phase space. So this is motion in the  $x, p$  plane. So let's think of the motion of this oscillator in the  $x, p$  plane. Here, we have that the energy is equal to  $p^2/2m + 1/2 m \omega^2 x^2$ . So this is an ellipse in the  $x$ -- a closed orbit. A constant  $\omega$  solution is an ellipse in this plane.

That's because it's some something squared plus something squared with different coefficient. So here it is. It's some sort of ellipse like that. Semi-major axis, semi-minor axis. I actually don't know which is the major and which is a minor. But two semi axes.

Well when  $p$  is equal to 0, what is the value of  $x$  defines this. So  $a$  is the value of  $x$  when  $p=0$ -- so it's  $\sqrt{2E/m\omega^2}$ . And  $b$  is the value of  $p$  when  $x$  is equal to 0. So it's just square root of  $2mE$ .

And here is the particle doing this motion in this orbit. As time goes by, the position  $x$  goes from a maximum to a minimum and return with the momentum going like that. It's a nice representation of the physical motion as moving on the ellipse. That's what this system is doing.

So you could ask, when you have something like that, you would ask, OK, you have an ellipse, what's the area of the ellipse? Area is  $\pi ab$ . That's the formula for the area from an ellipse.  $\pi$  times the product of the semi-major and semi-minor axes. That clearly generalizes correctly to a circle. And it's the right formula.

And then when we multiply it, look what happens. There is-- do I have-- yeah, I think I have everything here--  $\pi ab$ -- I get  $2\pi$ . The  $m$ 's cancel. The  $E$ 's don't cancel.  $E$  over  $\omega$ , hey, that's our adiabatic invariant.

$2\pi E/\omega$ , the area of this thing is our adiabatic invariant. That's a very nice classical picture. You have motion in phase space. And as  $\omega$  changes, maybe the ellipse will change. But the area tends to keep constant. That's what's happening. That's a statement of this result. So that's nice. It also can be written as a formula, which is kind of neat.

So the area of the ellipse is this. But the area of the ellipse, it can also be written in a slightly different way. Let's assume the orbit is going like this, for example. The motion is going like

that.

And then, what is the area of the ellipse? The area of this top part is the integral of  $p dx$ , is the integral of the top part. But we'll write it, that's just the top. But if I think of this as an integral over the whole counter, I would be having the whole integral of  $p dx$  like that into over the whole counter. I integrate here and I get the area of the top.

And when they integrate down here, I'm having the  $x$ 's that are negative and  $p$ 's that are negative. So I'm getting the area of the bottom part. So actually this full counter integral over the whole boundary gives you the full area of the thing. The top area in one part, the bottom area in the bottom part. So that's the area of the ellipse.

So we get the idea that the integral of  $p dx$  is roughly equal to  $2 \pi E$  over  $\omega$ . That's exactly equal when the system is time independent. But then if it's not time independent, this is an equation that can help us think of this system and identify an adiabatic invariant, because we identify this quantity as an adiabatic invariant. The more general statement in classical mechanics is that this kind of integral is an adiabatic invariant.

So in classical mechanics, people search for adiabatic invariants by integrals over phase space. It's a nice way to think of them. But let's go quantum mechanical. It's the analogies that we mentioned before. Let's use them.

So here we go. We've said a little about this. And we'll say a bit more.

So for quantum mechanics, what do we have? Well, we had the oscillator. We mentioned it. And we said that  $E$  over  $\omega$  was  $\hbar \omega$  occupation number plus  $1/2$  over  $\omega$  and is therefore  $\hbar n$  plus  $1/2$ .

So in quantum mechanics, the adiabatic invariant becomes a quantum number. And the adiabatic theorem in quantum mechanics is essentially going to say that if you have quantum numbers, you are almost guaranteed, if the system is slowly varying, to remain in that quantum state. We're going to try to make that clearer, but that's the spirit. Quantum numbers don't change under adiabatic approximation. Quantum numbers don't change.

So in some sense, this whole story was developed today. The classical intuition, maybe it's a little less obvious, is integrals over phase space, a trajectory in phase space of a particle conserves the area. And here, in quantum mechanics is the idea that if you have a quantum

number, you're going to find it difficult to have a change in quantum numbers.

But that's all we've been doing with time dependent perturbation theory, change of quantum number. So we'll think a little bit about it why that happened.

Now, there's more here that is interesting. You remember your WKB approximation. You did the quantization when you had a system, say, with two turning points,  $a$  and  $b$ . Bohr-Sommerfeld quantization, remember you assume there's a decaying thing here, therefore a cosine in the middle with a  $\pi/4$ , a decaying, another close  $n$  with a  $\pi/4$ . And the compatibility gave you the quantization condition.

Now, what was that quantization condition? It was  $1/\hbar$  integral of the local momentum  $p(x)$  from  $a$  to  $b$  equals  $n + 1/2$ . So if you multiply by 2 this integral, that is the full integral over the back and forth of  $p(x) dx$  is equal to  $2\pi\hbar(n + 1/2)$ . That formula, if you remember, gave the oscillator exactly.

But look how nice you see the intuition that this quantity of classical mechanics that we said is the idea adiabatic invariant also shows in semi-classical quantization telling you that that quantity, yes, doesn't want to change because, in fact, it represents a quantum number. So the WKB approximation is also reinforcing the idea that first this quantity, this integral over phase space is an adiabatic invariant and second that it represents a quantum number that doesn't want to change.

Let me make one last comment before we start a real calculation in quantum mechanics about this. Transitions, we studied transitions. And transitions are the kind of things that don't happen easily when you have an adiabatic approximation.

So what did we have for transitions? For transitions, we had the probability to go from some initial to some final state was the integral from 0 to  $t$  of  $e^{i\omega_{fi}t} \text{prime} \Delta H_{fi}(t) \text{prime} / \hbar v_{fi}$ .

Assume you have a constant perturbation. So you can take this quantity out of the integral,  $\Delta H_{fi} / \hbar^2$ . And you get this integral of  $e^{i\omega_{fi}t} \text{prime}$ . And that integral can roughly be done. And you've done it a few times--  $f_{fi}(t) \text{prime} / \omega_{fi}^2$ .

So this is what we got for a constant perturbation. And a constant perturbation finds it hard to induce energy jumping transition. So if you have a discrete system, making that transition is

hard, because however time you let go, this quantity is-- maybe there's a square here. Yes, there's a square-- this quantity is bounded in time. This doesn't grow beyond the particular quantity. But here you have a suppression because the energies are different. And if the energies are fairly different, this is very suppressed.

The way our calculation escape that, we'd said, oh, if you have a continuum behind these discrete states, then you can make a transition because you don't have to have a large change of energy. So our transitions are things that illustrate a little bit what we're getting to that it's difficult to change energy levels for slowly varying processes.

In fact, if this Hamiltonian was not exactly a time constant, it's still difficult to bank a transition, because, you know, if it varies slowly, this is still roughly true. Over some period of time, you could say, well, it's the average value.

In order to get an efficient transition between two energy levels, you had to put the Hamiltonian-- a cosine of  $\omega t$  at the right frequency. And then you induce the transition. But slowly varying Hamiltonian finds it difficult to induce transitions.

All right, so this is the end of our introduction to the subject of adiabatic evolution. And now we're going to try to calculate how a quantum state changes under adiabatic evolution.