## MITOCW | L14.2 Quantization of the magnetic field on a torus

**PROFESSOR:** One of the things you will be thinking about in the homework is the definition of gauge-invariant observables. So an operator O is a gauge-invariant observable-- gauge environment observable-- if, see, under a gauged transformation, O can change. The operator can change under a gauged transformation. You'd say, oh, but isn't it supposed to be gauge-invariant? But, well, it's not quite the operator itself that is gauge-invariant. It is its expectation value.

So you should have that psi prime of O prime psi prime is equal to psi O psi. So the operator may change. And that's OK.

The wave function will change as well. So the question is that if you put the new wave function and the new operator and you find the expectation value, do you get the same thing? That's what you really can ask. You cannot ask more than that.

And you will see very funny things when you do this. You will see that this operator, P, the momentum operator that you like, that you find has been intuitive for you so far, is not gauge-invariant. You cannot compute the expectation value of it and expect something physical. It's not gauge-invariant. The thing that is gauge-invariant is P minus q over c A-- is gauge-invariant.

So it's some small calculations you will do. In order to get the gauge-invariant operator, you do, in general, have to put in a dependence. That's the funny thing. So this thing will not be gauge-invariant. And, you know, it's almost obvious that P is not gauge-invariant. Why? Because P, under a gauge transformation-- well, the gauge transformation, say, change A, change phi prime, change psi. But it doesn't say change P. So in the gauge transformation, P goes to P.

But then let's look at psi prime P psi prime. Is it equal to psi P psi? Well, here you have that psi prime is psi times e to the minus i q lambda over h bar c. You have P here. And e to the i q lambda over h bar c. Psi prime-- no, psi. That's the left-hand side.

And this P is such that it takes derivatives. And lambda depends on space. So this is going to give you something. This is not going to cancel. This is not going to cancel, and therefore the left-hand side is not going to be equal to the right-hand side. The faces don't go across each other.

So therefore, there it is. P is no good. Not intuitive, not physical, not observable. It's a very strange thing that once you put electromagnetic fields, P, the canonical generator of translation, loses its privileged status. Not anymore we can think of P so easily.

And look, however. If you had here P minus qA, P minus qA is precisely the kind of thing that you can move the face across, and you will see that then it works. You get something to simplify, and it's all very nice. So gauge invariance makes for funny things to happen.

I want to do an example that illustrates quantization, another pretty surprising thing. So we have quite a few things to do today. So let me try to do this magnetic fields example. Magnetic field on a torus.

So we've talked about what is a circle. And we say a circle is the line with the identification x equal x plus L. So the point 0 is identified with the point L. And that's a circle. This line on this point is the same as this one, so you return.

A torus that we use-- so, this is a circle-- a torus, which is something that we sometimes think of like this, is, on the other hand, an identification of the following kind. Here is the x-axis. Here is the y-axis. Here is Lx. Here is Ly. And we say that any point xy is identified with x plus Ly Lx, and every point xy is identified with xy plus Ly, which is to say that any point with some value of x is identified with a point with x increasing by Lx. So this line is supposed to be glued to this line. And this line is supposed to be glued to that line. So this is identified and this is identified.

Perhaps the most intuitive way of thinking of this is take this strip, take these two sides, glue them. Now you have a cylinder. And now you're supposed to glue this end to the first end. So you glue them like that, and you form something like this.

So here is a torus. Now we try to put the magnetic field on this torus. So you could say, well, this is x and y. There is a torus. There's the z-direction. Let's put the magnetic field that goes through the torus in the z-direction. B z. B in the z-direction.

And we'll put a constant one, B constant. So it's time-independent and space-independent. And then you say, OK, let's look at my Maxwell's equations. Divergence of B equals 0. B is constant. Good. Curl of B is related to current. There's no current. Plus dE dt, displacement current. There's no electric field. Good equation. Curl of E is minus dB dt, no E.

There's no problem, obviously. So B, for any constant value of B, it satisfies Maxwell's equations. So any constant value of B should be an allowed magnetic field. Or so we would think, because it actually is not.

So why does it go wrong, this intuition? We'll see. But let's put an assumption. The assumption is that we're doing quantum mechanics, and there exists a particle with charge. And I'll call it this time q. I'll just--

So if we're doing quantum mechanics, and there exists a particle of charge q, I need potentials to describe the quantum mechanics. I need an A field to describe this. So I know B is consistent, and it makes sense, and we can use it, if and only if I can find a vector potential. So our task is to find a vector potential on this torus.

Well, at first sight that doesn't sound too difficult. You can say, all right, let's find the vector potential. That's easy to find an A such that curl of A is equal to B. So B z is dx A y minus dy A x.

So let's simplify my life. Let's take A y to be equal to B z, and let's call this magnetic field B0, the constant value B0. Constant. So A y would be equal to B0 times x, and A x would be 0.

OK. That's it. Look. A y is equal to B0 x. So I take the derivative dx of A y. I get B0. dy of A x is 0. Perfect. All done, we would say. You found the A y. But there is a problem with this [? A i. ?] What is the problem?

If this is a torus, it means these points are the same as these points. So I should have, the vector potential here must be the same as the vector potential there. Because it is, after all, the same point. You're gluing the surface.

But, OK, this point and this point differ by y. And this vector potential doesn't depend on y. So, phew. That's good. A is well-defined because it has the same value anywhere here as anywhere there. So I can write it as A y of x, y plus L y at any point x. And y plus L y is the same thing as A y of xy. So that's very good.

But we're going to run into trouble in the second now, because the A y should be the same here and here, too. Because that's also the same points in the torus. This line is identified with this. So when I change x by L x, A y should not change. And here it seems like A y at x plus L x, and the same point y, is not the same as A y at xy.

And this is a torus, so this point, x plus L x and y and y should be the same. So this vector potential is not the same here as here. So, actually, this vector potential doesn't look good. You thought you could write the magnetic field in terms of a vector potential, but this vector potential doesn't seem to live on the torus. It's just not a well-defined magnetic vector potential in the torus. It doesn't have the same value here and here.

So in principle, we're not through. We're not out of the woods here. We have not been able to find a good vector potential yet.

But here, the gauge transformations come a little to the rescue. When you work with vector potentials, you don't really need that the vector potential be the same here as here. It is enough if the vector potential here and here, which is the same point, they differ by a gauge transformation.

So the vector potential is a subtle object. Here is the torus. In one part of the torus it has a formula. In the other part, it may look like it's not consistent. But you can use another formula related by a gauge transformation. So this vector potential here is OK if I manage to show that what I get on this side is just the gauge transformation of what I was getting here. So physically these are the same. And yes, you have a unique configuration on the torus if the vector potential here is gauge-equivalent to the vector potential there.

So let's try to do that.

So, gauge transformations to the rescue. OK. A y of x plus Lxy. I want it to be a gauge transformation of the field at A y xy. So remember, A prime was A plus gradient of lambda. So I'm going to write this as A at the same point, which is xy-- it's physically the same point-- plus the gradient of lambda, in this case would be dy lambda.

OK. Now we have to find the gauge parameter. To show that they're gauge-equivalent here and here, I must find the gauge parameter. That's the lambda. So let's do the formula here. This is B0. There's a formula here, B0 times x. So x plus Lx times x plus Lx is equal to B0 times x plus dy lambda. So I cancel here and I get B0 Lx is equal to dy lambda. OK.

All right. So, well, it seems that we're succeeding again, against all odds. And we could take lambda, for example, to be B0 Lx times y. And that would be our gauge parameter.

OK. So are we happy already? Have we shown that everything is working? Well, there's still a problem. This time the gauge parameter doesn't seem to be well-defined on the torus as well when I change-- you know, we didn't have problems with y here. We had problems with x. When we tried to fix it with x, we find the gauge parameter. But the gauge parameter doesn't seem to have-- it does have problems with y. It's not the same at this point and at this point. Because at those points, y differs by L y, and the gauge parameter, again, doesn't quite seem to live on the torus.

It seems like an endless amount of complication. But we're near the end of the road now. The thing that comes to our rescue is that lambda, in fact, doesn't quite have to be that well-defined. Why? Because the gauge transformation of the charged particle says that you need to do i Q over hc lambda times psi. That's how you do gauge transformations.

And in fact, this, remember, this is the U thing. And even though we write A prime as A D lambda, this term can be written in terms of U. U is the master quantity that has all the information. In fact, D lambda is roughly U minus 1 gradient of U, up to a series of factors.

So everything depends on U. So we don't have to worry too much if lambda is not well-defined on the torus. What has to be well-defined on the torus is U, the exponential of lambda. And that, it's a saver, because U, now, is e to the i q. Lambda is B0 Lxy over h bar c. That's U. And U will be well-defined on the torus if, when you change y plus y plus L y, this doesn't change.

And for that, this whole thing, when you change y by L y, must change by a factor of 2 pi. So if-- you see, you're living dangerously with this. It just doesn't want to exist on a torus. But if this happens to be equal to 2 pi n-- so if you have q B0 Lx Ly over hc equal to 2 pi n, everything is OK. That thing, when you change y for y plus L y, this whole exponent changes by 2 pi n. And i times 2 pi n is 1. So U is well-defined on the torus.

So we're almost there. Here it is. B0-- there is a quantization. B0 times the area of the torus is equal to 2 pi n-- pi n-- hc over q n.

So you cannot have arbitrary magnetic field. The magnetic field is such that the flux is quantized. So here is the illustration of an example I wanted to mention from the beginning, that if you have a magnetic field that solves Maxwell's equations, you're not done. You have to find potentials. And sometimes there are funny things happening. And in particular, here is the funny thing that has happened here, is that in order to have a well-defined vector potential on the torus, you've been forced to quantize the magnetic field.

And B0 times A is equal to phi, the flux. And this flux is a multiple of this quantity, which is sometimes called the quantum of flux, the least possible flux. n, the flux quantum.

So this flux quantum, it is a famous number, in fact. I have it somewhere here.

Somewhere in my notes. Yes, here it is. Phi 0 for electrons, for q equal to e-- phi 0-- these are units you seldom use. But it's about 2.068 times 10 to the minus 15 webers. Who knows what a weber is?

Or 2.068 times 10 to the minus 7 Maxwells. Anybody know what a Maxwell is?

OK, a weber is a unit of flux. It's tesla times meter squared. That's far too big. The Maxwell, it's a more natural thing. Maxwell is gauss times centimeters squared.

So, you know, the magnetic field of the Earth is about half a gauss. A centimeter square you can imagine. And that's the value of the flux quantum, which will have a role later for us.