PROFESSOR: Today, we have to discuss harmonic perturbations. So we've done Fermi's golden rule for constant transitions. We saw transitions from a discrete state to a continuum. And by integrating over the continuum, we found a nice rule, Fermi's golden rule, that govern the transition rate for this process.

So the only thing we have to do different now is consider the case that the perturbation is not just a step that gets up and stays there, but it has a frequency dependence. So that will bring a couple of novel features. But at the end of the day, as we will see, our Fermi's golden rule is going to look pretty similar to the original Fermi's golden rule.

A nice application of Fermi's golden rule is the calculation of the ionization rate for hydrogen, in which you take a hydrogen atom, you put it in an electric field or send a light wave, and then suddenly the electron and the hydrogen atom from the ground state ionizes. And we can compute already-- we have the technology to compute the ionization rate. That's a pretty physical quantity. And that will be an example we'll develop today.

Those rates have the funny situation that the calculation can be somewhat involved and interesting. And the answers, generally, by the time you simplify them, are pretty simple and pretty nice. So it's a good idea. You have to have patience with those calculations to simplify it till the end, and that's pretty instructive.

So we begin with harmonic perturbations. So we did constant perturbations already. So now harmonic perturbations. So our situation is that in which age H of t is equal to a known Hamiltonian plus delta H of t. And this time, delta H of t is conventionally written as 2 H prime cosine omega t for some t between t0 and 0, and 0 otherwise.

All of us wonder why the 2 here. One reason for it-- it's all convention, of course. You have your perturbation, and what you call E H prime or what you call 2 H prime is your choice. But this 2 has the advantage that when you describe the cosine in terms of exponentials-- e to the i omega t plus e to the minus i omega t-- the over 2, it cancels this one.

And that makes Fermi's golden rule, that will follow also and will be valid for these perturbations, take exactly the same form as it did for the case of constant perturbation. So it's fairly convenient to put that 2, and we'll put it in. Some books don't, and then they have different looking formulas for Fermi golden rule depending to which case you're talking about.

Of course, when we mean that this is the time dependence, we are implying that H prime is time independent. Because the time dependence is this one. That's what we're interested in considering. Of course-- this has been asked sometimes-- H prime can depend on all kinds of other thing-- position coordinates, some other quantities. But we're focusing on time here, so we'll leave it there.

Moreover, for reasons of convention, just let's always thing of omega as positive. It wouldn't make a difference if it would be negative here with the cosine function, but let's just set by convention that omega is positive.

Finally, we're going to do transitions again from an initial to a final state. So we will consider the case when we go from an initial state to a final state. And therefore, we will work in this language with this constant coefficients Cn's, these coefficients that multiply the states in psi tilde. Psi tilde is equal to Cn n of t, sum over n. And these Cn's, at time equals 0, will be equal to delta ni, which means that they are all 0 except when we're talking about Ci at 0 is equal to 1 because we start with an initial state.

We had a general formula for the transition coefficient. And Cm of 1 at time equal t-- or I'll put t0-- is equal to sum over n, integral from 0 to t0 e to the i omega mn t prime, delta H mn of t prime over i h bar, Cn at time equals 0, dt prime. This was our general formula for transition coefficients.

The Cm's is the coefficient or the amplitude for the state to be found in the m eigenstate at time t0 to first order in perturbation theory. And it depends on where you started on. That's why the sum over n here with initial states. But this sum is going to collapse because we know we start with the state i. So when we substitute Cn equal to this, the sum only works when n is equal to i. So we'll put for ni's.

And, of course, we're going to also take for the final state to be f. So the formula now reads Cf 1 at t0 is equal to integral from 0 to t0 e to the I omega f-- m was f-- i t prime. And now the delta H. The delta H is this whole quantity, so we have to substitute it.

So 2 H prime is the only part that has matrix elements. The cosine omega t is just a function. So it's H prime fi cosine omega t prime, and dt prime. There's the i h bar.

So I think I got everything right. The sum collapsed. mn is being replaced by the right labels. mn here, this is the expectation value between m and n. And that becomes between f and i.

And it affects this whole thing, but it just ends up affecting the Hamiltonian H prime here.

So I think we're OK. We have everything there. And Hfi, of course, doesn't have time dependence. So we said H prime has no time dependence. So that thing can go out of the integral. So this will go out.

And the integral is simple because you have now Hfi prime over i h bar. And the 2, we leave it for the cosine. So we get two integrals. t0 e to the i omega fi plus omega t prime-- from the first exponential in the cosine-- plus an e to the i omega fi minus omega t prime-- from the second exponential in cosine-- dt prime.

Well, that's very nice. This is all doable. The Hfi doesn't give us any trouble. It's a constant. It's out of the integral. It's all pretty nice and simple.

So we can do these two integrals. They're integrals of exponentials, so it's just an exponential divided by those coefficients. So I'll just do it and evaluated it between t0 and 0.

So what do we get? Minus i Hfi prime over h bar, e to the i omega fi plus omega t0, minus 1, over omega fi plus omega. You can imagine that e to the i omega t integrates to e to the i omega t over omega. So that's why that works. And the two limits are t0 and 0.

Plus e to the i omega fi minus omega t0, minus 1 again, over omega fi minus omega. Great. Our integral is there. It's done. And now it's time to appreciate what it tells us, because it tells us something very important, this formula.

So you look at this and you say, well, OK. This is the transition amplitude to state omega-- I'm sorry-- state f, final state. And it depends on omega fi. And omega fi just Ef minus Ei over h bar. So if I know my final discrete state Ef, I can figure out what is the transition probability.

Now, these denominators are intriguing because maybe if you adjust the frequency omega-- suppose you have an initial state and a final state here. You may adjust the frequency omega to match them, and in that case maybe make the denominators equal to 0. And that's exactly what kind of happens here.

So, first of all, if you look at this expression, it's a sum of two terms that are added together and multiplied by a constant. As t0 really goes to 0, completely goes to 0, t0, each factor, actually, if you see the Taylor expansion of the exponential, you cancel the 1 and you then cancel the linear term with the denominator. This is just i t0. And this is also i t0. So they're comparable as time is really going to 0.

But time going to 0 is never of interest for us. For us, we need to be the time a little big already so that our calculations, as we did in the constant transitions that had lobes that decreases constants over t0-- we needed the time to be sufficiently large so that the lobes are narrow. And that, we could guarantee.

So t0 going to 0 is not very interesting. We need t0 a little bigger. Not too big that the rate of a process overwhelms the probability. But we need a little big.

So, in that case, the numerators are going to be bounded numbers. You see, you have an exponential minus 1. So that varies from-- the magnitude of this thing varies from 2 to 0, basically. In fact, in these numerators, you can see, if the phase is 0 for some particular value-- if the exponential has a phase that's proportional to 2 pi, this is 0. And then sometimes this exponential is minus 1, so it gets to minus 2. So it's finite.

And the same is here. So these are bounded numerators. On the other hand, you may have the possibility that these things become 0. And those are the cases that are of interest to us, the cases when those terms are going to be 0.