PROFESSOR: You first are facing the calculation of the energy eigenstate with some arbitrary potential. You probably want to know some of the key features of the wave functions you're going to calculate. So in fact, all of today's lecture is going to be devoted to this intuitive, qualitative insights into the nature of the wave function. So we will discuss a few properties that help us think clearly. And these are two of those properties. I want to begin with them. Then we'll do a third one that we have already used, and we will prove it completely. And then turn to the classical and semi-classical intuition that lets us figure out how the wave function will look.

And that's a great help for you. Even if you're solving for your wave function numerically, you always need to know what the answer should look like. And it's ideal if before you calculate, you think about it. And you realize, well, it should have this t properties. And if you find out that those are not true, well, you will learn something about your intuition and see what was wrong with it.

So we're talking about one dimensional potentials, time independent potentials. And a first statement that is very important, and you will prove in an exercise after spring break, and that is the fact that one dimensional potentials, when you look at what are called bound states, you never find degeneracies, energy degeneracies. And this is when x extends from minus infinity to infinity. You've seen already, in the case of a particle in a circle, there are degenerate energy eigenstates. But if the potential extends to infinity, there is no such things.

Now what is a bound state? A bound state sounds like a complicated concept. But it is not. It really means an energy eigenstate that can be normalized. Now if an energy eigenstate can be normalized and you live in the full real line, that the wave function must go to 0 at infinity. Otherwise you would never be able to normalize it. And if the wave function goes 0 at infinity, the bound state is some sort of bump in the middle region or something like that. And it eventually decays. So this is bound by the potential in some way. And that's basically what we use to define a bound state. We'll take it to be that generally. So this is something, this property, which is very important, is something you will prove.

But now we go to another property. We've emphasized forever that the Schrodinger equation is an equation with complex numbers. And the solutions have complex numbers. And suddenly, I wrote a few lectures ago a wave function was real. And I was asked, well, how can it be real? Well, we've discussed stationary states in which the full wave function, capital PSI, is

equal to a little psi of x times the exponential of e to the minus i et over h bar. And there in that exponential, there is complex numbers on this little psi of x in front of that exponential, which is what we called basically those energy eigenstates. The e to the minus i et over h bar, it's understood that little psi of x is the thing we've been looking for. And this psi of x solves the time independent Schrodinger equation h psi equal e psi. And that equation has no complex number in it.

So little psi of x can be real. And there's no contradiction. Because the full solution to the time dependent Schrodinger equation is complex. But here is a statement. With v of x real, the energy eigenstates can be chosen to be real. And the words can be chosen are very important here. It means that you may find a solution that is complex, but you need not stick to that solution. There is always a possibility to work with real solutions.

And what is the way you prove this? This I will put this in the notes. You don't have to worry about the proof. You consider the Schrodinger equation for psi. And you show that psi star, the complex conjugate of psi, solves the same equation that psi solves. And therefore, if psi is a solution, psi star is a solution with the same energy. That part is very important. Therefore, if you have two energy eigenstates with the same energy, you can form the sum. That's still an energy eigenstate with the same energy. Even formed in difference, that's still an energy eigenstate with the same energy. And the sum of psi plus psi star is real. And the difference psi minus psi star, if you divide by 2i, is real as well. Therefore you can construct two solutions, the real part of psi and the imaginary part of psi. And both are solutions to the Schrodinger equation.

So I've said in words what is the proof of the first line. It's that if you have a psi, psi star is also a solution. Therefore, psi plus psi star and psi minus psi star are solutions. So given a complex psi, then psi psi of x. Then psi real of x that we define to be psi of x plus psi star of x over 2. And the imaginary part of the wave function 1 over 2i psi of x minus psi star of x are both solutions with the same energy as this one has. So these are the two solutions. So far so good. You don't like to work with complex psi? No need to work with complex psi. Work with real psi.

But here comes the second part of the argument, the second sentence. I want you to be alert that the second sentence is very powerful. It says that if you have a bound state of a one dimensional potential, more is true. There are no genuinely complex solutions in this case. Any solution that you will find, it's not that it's complex and then you can find the real and imaginary

part. No, any solution that you will find will be basically real. And how can it fail to be real? It just has a complex number in front of it that you can ignore. So it is a very strong statement. That the wave function, it's not that you can choose to work it. You're forced to do it up to a phase.

So how is that possible? How is that true? And here is the argument for the second line. If we're talking bound states, then these two are real solutions with the same energy. So now suppose these are bound states. There is a problem if there are two real solutions with the same energy. They would be degenerate. And property number 1 says there's no such thing as degenerate energy bound states. So they cannot be degenerate.

So if you start with a complex psi, and you build these two, they must be the same solution. Because since there are no degenerate bound states, then psi, I will write it as psi imaginary, of x must be proportional to psi real of x. And both are real, so the only possibility is that they are equal up to a constant, where the constant is a real constant.

You see there cannot be degenerate bound states. So the two tentative solutions must be the same. But that means that the original solution, psi, which is by definition the real part plus i times the imaginary part, is now equal to psi r plus i times c times psi r again, which is 1 plus ic times psi r. And that is basically the content of the theorem. Any solution is up to a number, just the real solution. So you're not going to find the real solution has non-trivial different real imaginary parts here. No, just the real solution and a complex number.

Now if you want, you can just write this as e to the i argument of 1 plus ic times square root of 1 plus c squared psi r. And then it's literally the way it's said here. The wave function is proportional to a real wave function up to a phase. So that's a very neat situation. And therefore, you should not be worried that we are going to have to assume many times in our analysis that the bound states were trying to look for are real. And we plot real bound states. And we don't have to worry about, what are you plotting? The real part? The imaginary part? Many times we can just work with real things.