

**PROFESSOR:** Simple harmonic oscillator.

So what is there about this simple harmonic oscillator? Well, it's a classical system that you understand perfectly well. An oscillator, a spring with a mass oscillates and has an energy, which is the kinetic energy plus the potential energy, and that's  $p^2$  over  $2m$  plus  $\frac{1}{2} m \omega^2 x^2$ . And this may be a tiny bit unfamiliar, this way of writing it. But you may recall that  $\omega$  is equal to the square root of  $k$  over  $m$ , the so-called spring constant, in which the potential in terms of  $k$  would be  $\frac{1}{2} k x^2$ . And that's the potential energy stored in a spring that you stretch at distance  $x$ .

That's the total energy of a harmonic oscillator. So when physicists starting with quantum mechanics in the '20s decided, let's do a harmonic oscillator, a quantum harmonic oscillator, they had to invent the Hamiltonian. And the Hamiltonian they invented was a simple one. They looked at that and said,  $\hat{H}$  is going to be  $\hat{p}^2$  over  $2m$  plus  $\frac{1}{2} m \omega^2 \hat{x}^2$ . And now the difference is going to be that  $\hat{x}$  with  $\hat{p}$  are operators, and this is our  $\hbar$ , and that's going to be my quantum system.

So this is a quantum system that is inspired by classical mechanics in the purest and simple way. Anyone could have invented this quantum system. It was very natural.

Still, the result of the quantization is very surprising because while this mechanical oscillator can oscillate with any amplitude, the quantum oscillator has quantized amplitudes and quantized energies therefore. So all kinds of interesting things happen with this oscillator.

Now the reason this is also very ubiquitous is that this potential is exactly a quadratic potential  $V$  of  $x$ .  $V$  of  $x$  is  $\frac{1}{2} m \omega^2 x^2$ . We have  $x$  without the hat. This is a good approximation to almost any system we consider in nature, any oscillating system. Because for any potential that has a minimum, there is some parabolic approximation at the bottom. At the bottom the derivative vanishes, so the Taylor series says that approximately at the bottom is a quadratic potential.

And therefore this quadratic potential will govern the quantum oscillations of a diatomic molecule, the quantum oscillations of a periodic system, all kind of quantum oscillations will be approximately governed by a harmonic oscillator. Light has a harmonic oscillator description for its photons.

This Hamiltonian is the most famous Hamiltonian there is. In fact, when you have electrons in a magnetic field, somehow this shows up. And this becomes some sort of problem that you solve very well, understand very well, and suddenly it pops up in all kinds of contexts. So we need to understand it.

And let's go directly to the issue of solving this problem, because it has many important lessons. So there's two ways to solve this problem of finding the bound states of the energy eigenstates of the harmonic oscillator.

This is a very interesting potential because all its energy eigenstates are bound states. That's not the case for the delta function potential. In the delta function potential we found one bound state, but they're all kind of unbound states with positive energy. But this potential grows forever, never stops growing. So whatever energy you have, it is a bound state. It will decay. It will be localized. So you just have bound states. It's marvelously nice because of that property. Much simpler than anything you can imagine.

So what do we have to do? We want to find the energy eigenstates. So we'll write  $\hat{H}$ , and I will write  $\psi$ . People write sometimes  $\phi$ .  $\hat{H} \psi_n = E_n \psi_n$ . And we don't know the energy eigenstates. You know it's a symmetric potential. It's a real potential. This we used to go  $\psi_n$ 's, but I will write them as  $\phi_n$ 's as many people do, because they are the harmonic oscillator ones that are very famous. And we don't know-- this could be  $E_n$ , energy of the  $n$ -th state. And we don't know what are the energies nor how the wave functions look. And we have to solve a differential equation.

As I was saying, there's two ways of solving this differential equation. One, treating this as a differential equation and understanding why the energy is quantized from the differential equation. This actually gives you a lot of insight as to what's going on, and it will relate to the kind of things you're doing in the homework this week.

The other way is to be very clever and invent what are called raising and lowering operators, and sort of solve this whole system without solving the general differential equation, by solving a first order very simple differential equation, and then doing everything else algebraically with creation and annihilation operators. We will also do that. But we will not develop that too far. That's The applications of that method are mostly for 805. So we'll introduce creation and annihilation operators, which are very nice and very useful. But we leave some of the applications to coherent states of harmonic oscillators, to squeezed states of harmonic

oscillators, for later.

So this is what we want to solve. So we have minus  $\hbar^2$  over  $2m$   $d^2 \psi / dx^2$ . And I will probably forget about the labels. The labels will come later as we solve the equation. Plus  $\frac{1}{2} m \omega^2 x^2 \psi$ , is equal to  $E \psi$ .

OK. This is the question we want to solve. Now we're going to do one thing first with this differential equation. We don't like all these dimensionful constants. If you had to put it in the computer, what? Are you going to put  $6 \times 10^{-23}$  over  $m$ ? And you won't solve a differential equation. We have to clean this up.

To clean this up, there is a procedure that is guaranteed to do the job for you. And the procedure is to change the  $x$  variable into a variable that has no units. This is guaranteed to lead you to the way to solve this differential equation. You will be using this throughout the semester. Cleaning differential equations is a nice skill.

And the fact is that there's a method, and the method always proceeds by first writing  $x$  equals  $au$ , where this will be unit-free, this quantity  $u$ . So this will become a differential equation on a unit-free constant, which is ideal for your numerical solution and is much nicer.

But then this  $a$  to have units of length. So the first thing you have to do is look up in your problem. What are your constants? And you have a mass  $m$ , a frequency  $\omega$ , an  $\hbar$ . And you need to find a constant with units of length. Wasn't that in your test? I think so.

How do you find a constant with units of length here? Well, energy is equal to-- you can write it in two ways. The energy can be written as  $p^2 / m$ , so it will be  $\hbar^2$  over  $m$  times a length squared. But from the potential, it also has the units of  $m \omega^2 a^2$ . These are units, equations for units. The units of energy are these, and the units of energy from the second term in the Hamiltonian are those. From where you get that  $a^2$  is equal to  $\hbar / m \omega$ .

So that's the constant that you need. If you have that constant, your differential equation becomes what? Well, it becomes the following. Let's write it out. It's very simple, because  $x$  is equal to  $au$ , and therefore you get minus  $\hbar^2$  over  $2m a^2 d^2 \psi / du^2$ .  $x$  is equal to  $au$ , so it basically just shows up here. Plus  $\frac{1}{2} m \omega^2 a^2 u^2 \psi$ , is equal to  $E \psi$ .

Now things have to work nicely. If you did the job well, they have to work nicely. Let's think of

the units of this equation. Phi is here, phi is here, phi is here, so phi is not an issue. The units of phi are irrelevant. Here is units of energy, but u has no units. This has no units. So this must have units of energy. And since u has no units, this derivative has no units, and this must have units of energy. And these two numbers must be something nice if you substitute a squared.

And indeed, if you substitute a squared here, this whole number becomes  $\hbar \omega$ . And this number becomes  $\hbar \omega$  as well, which is very nice. So this whole equation has become  $-\frac{1}{2} \hbar \omega \frac{d^2 \phi}{dx^2} + \frac{1}{2} \hbar \omega u^2 \phi = E \phi$ .

The next step is to say, you know, I don't even want these energy units. Even though they don't look that bad, this equation looks much nicer than the original equation which had all kinds of strange units. So I will multiply this equation by  $\frac{2}{\hbar \omega}$  to cancel it. So I get  $-\frac{d^2 \phi}{dx^2} + u^2 \phi = \frac{2E}{\hbar \omega} \phi$ .

So look at this. The equation is now almost in perfect form. And in order to make it perfect, I would say that the right hand side-- now, see again. Phi, whatever units it has, it doesn't matter. It's all over the place. But this has no units, this derivative, and this has no units, this multiplication. So this must have no units. And indeed, you know that  $\hbar \omega$  has units of energy, and that's energy.

So it suggests that you define a unit-free energy, free energy, which is  $\frac{2E}{\hbar \omega}$ . And calculating curly E is the same thing as calculating the energy, because if you know this number you know the energy. The energy is  $\frac{\hbar \omega}{2}$  times curly E. The advantage is that curly E will be either 1, 2, 1/2, a nice number, while E is some 0.87 eV, or things like that.

So all of this equation has been reduced to this very nice equation.  $-\frac{d^2 \phi}{dx^2} + u^2 \phi = E \phi$ . Or,  $\frac{d^2 \phi}{dx^2} = u^2 \phi - E \phi$ .