PROFESSOR: Today we'll talk about observables and Hermitian operators. So we've said that an operator, Q, is Hermitian in the language that we've been working so far, if you find that the integral, dx psi 1 Q psi 2, is actually equal to the integral dx of Q, acting this time of Psi 1 all star psi 2.

So as you've learned already, this requires some properties about the way functions far away, at infinity, some integration by parts, some things to manage, but this is the general statement for a large class of functions, this should be true. Now we want to, sometimes, use a briefer notation for all of this. And I will sometimes use it, sometimes not, and you do whatever you feel. If you like to use this notation, us it. So here's the definition. If you put up Psi 1, Psi 2 and a parentheses, this denotes a number, and in fact denotes the integral of psi 1 star of x, psi 2 of x dx.

So whatever you put in the first input ends up complex motivated. When you put in the second input, it's like that, it's all integrated. This has a couple of obvious properties. If you put a number times psi 1 times psi 2 like this, the number will appear, together with psi 1, and will complex conjugated. So it can go out as a star psi 1 psi 2. And if you put the number on the second input, it comes out as is. Because the second input is not complex conjugated in the definition. With this definition, a Hermitian operator, Q is Hermitian, has a nice look to it. It becomes kind of natural and simple.

It's the statement that if you have psi 1, Q psi 2, you can put the Q in the first input. Q psi 1 psi 2. This second term in the right hand side is exactly this integral here. And the first tern in the left hand side is the left hand side of that condition. So it's just maybe a briefer way to write it. So when you get tired of writing integral dx of the first, the second, you can use this.

Now with distance last time, the expectation values of operators. So what's the expectation value of Q in some state psi of x? And that is denoted as these braces here and of psi is equal to the integral of psi. The expectation value depends on the state you live in and it's psi Q psi. Or if you wish, dx in written notation psi Q. I should put the hats everywhere. This is the expectation value of Q. I'm sorry, I missed here a star. So so far, so good. We've reviewed what a Hermitian operator is, what an expectation value is, so let's begin with some claims.

Claim number one. The expectation value of Q, with Q Hermitian. So everywhere here, Q will be Hermitian. The expectation value of Q is real. A real number, it belongs to the real numbers. So that's an important thing. You want to figure out the expectation value of Q, you have a psi star, you have a psi. Well, it'd better be real if we're going to think, and that's the goal of this discussion, that Hermitian operators are the things you can measure in quantum mechanics, so this better be real.

So let's see what this is. Well, Q psi, that's the expectation value. If I complex conjugate it, I must complex conjugate this whole thing. Now if you want to complex conjugate an integral, you can complex conjugate the integrand. Here it is. I took this right hand side here, the integrand. I copied it, and now I complex conjugated it. That's what you mean by complex conjugating an integral. But this is equal, integral dx. Now I have a product of two functions here. Psi star and Q that has acted on psi. So that's how I think. I never think of conjugating Q. Q is a set of operations that have acted on psi and I'm just going to conjugate it. And the nice thing is that you never have to think of what is Q star, there's no meaning for it.

So what happens here? Priority of two functions, the complex conjugate of the first-- now if you [INAUDIBLE] normally something twice, you get the function back. And here you've got Q psi star. But that, these are functions. You can move around. So this Q hat psi star Q psi. And so far so good. You know, I've done everything I could have done. They told to come to complex conjugate this, so I complex conjugated it and I'm still not there. But I haven't used that this operator is Hermitian. So because the operator is Hermitian, now you can move the Q from this first input to the second one. So it's equal to integral dx psi star Q psi. And oh, that was the expectation value of Q on psi, so the star of this number is equal to the number itself, and that proves the claim, Q is real.

So this is our first claim. The second claim that is equally important, claim two. The eigenvalues of the operator Q are real. So what are the eigenvalues of Q? Well you've learned, with the momentum operator, eigenvalues or eigenfunctions of an operator are those special functions that the operator acts on them and gives you a number called the eigenvalue times that function. So Q, say, times, psi 1, if psi 1 is a particularly nice choice, then it will be equal to some number. Let me quote Q1 times psi1. And there, I will say that Q1 is the eigenvalue. That's the definition. And psi1 is the eigenvector, or the eigenfunction. And the claim is that that number is going to real.

So why would that be the case? Well, we can prove it in many ways, but we can prove it kind of easily with claim number one. And actually gain a little insight, cold calculate the expectation value of Q on that precise state, psi 1. Let's see how much is it. You see, psi 1 is a particular state. We've called it an eigenstate of the operator. Now you can ask, suppose you live in psi 1? That's who you are, that's your state. What is the expectation value of this operator? So we'll learn more about this question later, but we can just do it, it's the integral of dx psi 1 Q psi 1. And I keep forgetting these stars, but I remember them after a little while. So at this moment, we can use the eigenvalue condition, this condition here, that this is equal to dx psi 1 star Q1 psi 1. And the Q1 can go out, hence Q 1 integral dx of psi 1 star psi 1.

But now, we've proven, in claim number one, that the expectation value of Q is always real, whatever state you take. So it must be real if you take it on the state psi 1. And if the expectation value of psi 1 is real, then this quantity, which is equal to that expectation value, must be real. This quantity is the product of two factors. A real factor here-- that integral is not only real, it's even positive-- times Q1. So if this is real, then because this part is real, the other number must be real. Therefore, Q1 is real.

Now it's an interesting observation that if your eigenstate, eigenfunction is a normalized eigenfunction, look at the eigenfunction equation. It doesn't depend on what precise psi 1 you have, because if you put psi 1 or you put twice psi 1, this equation still holds. So if it hold for psi 1, if psi 1 is called an ideal function, 3 psi 1, 5 psi 1, minus psi 1 are all eigenfunctions.

Properly speaking in mathematics, one says that the eigenfunction is the subspace generated by this thing, by multiplication. Because everything is accepted. But when we talk about the particle maybe being in the state of psi 1, we would want to normalize it, to make psi 1 integral squared equal to 1. In that case, you would obtain that the expectation value of the operator on that state is precisely the eigenvalue. When you keep measuring this operator, this state, you keep getting the eigenvalue. So I'll think about the common for a normalized psi 1 as a true state that you use for expectation values.

In fact, whenever we compute expectation values, here is probably a very important thing. Whenever you compute an expectation value, you'd better normalize the state, because otherwise, think of the expectation value. If you don't normalize the state, you the calculation and you get some answer, but your friend uses a wave function three times yours and your friend gets now nine times your answer. So for this to be a well-defined calculation, the state must be normalized.

So here, we should really say that the state is normalized. Say one is the ideal function normalized. And this integral would be equal to Q1 belonging to the reals. And Q1 is real. So for a normalized psi 1 or how it should be, the expectation value of Q on that eigenstate is precisely equal to the eigenvalue.