8.04: Quantum Mechanics | Professor Allan Adams
Massachusetts Institute of Technology | March 14 2013

Exam 1

Last Name: ______________________________

First Name: ______________________________

Check	Recitation	Instructor	Time
	R01	Barton Zwiebach	10:00
	R02	Barton Zwiebach	11:00
	R03	Matthew Evans	3:00
	R04	Matthew Evans	4:00

Instructions:
Show all work – No scratch paper. All work must be done in this exam packet.
This is a closed book exam – books, notes, phones, calculators etc are **not allowed**.
You have 50 minutes to solve the problems. Exams will be collected at 12:00pm sharp.

Problem	Max Points	Score	Grader
1	80		
2	20		
Total	100		

Formula Sheet 1

Fourier Transform Conventions:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; e^{ikx} \tilde{f}(k) \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; e^{-ikx} f(x)$$

Delta Functions:

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x-a) = f(a) \qquad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}$$

$$\delta(x) = \left\{ \begin{array}{cc} 0 & x \neq 0 \\ \infty & x = 0 \end{array} \right\} \qquad \delta_{mn} = \left\{ \begin{array}{cc} 0 & m \neq n \\ 1 & m = n \end{array} \right\}$$

Operators and the Schrödinger Equation:

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} \qquad [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \qquad i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{E}\, \psi(x,t)$$

$$\hat{E} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \qquad E\, \phi_E(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \phi_E(x) \; + \; V(x)\, \phi_E(x)$$

Common Integrals:

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \qquad (f|g) = \int_{-\infty}^{\infty} dx\, f(x)^*\, g(x)$$

For an infinite square well with $0 \leq x \leq L$:

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin{(k_n x)} \qquad (\phi_n|\phi_m) = \delta_{mn}$$

$$k_n = \frac{(n+1)\pi}{L} \qquad E_n = \frac{\hbar^2 k_n^2}{2m}$$

Formula Sheet 2

Raising and Lowering Operators for the 1d Harmonic Oscillator ($\beta^2 = \hbar/m\omega$):

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{1}{\beta}\hat{x} + i\frac{\beta}{\hbar}\hat{p}\right) \ , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\beta}\hat{x} - i\frac{\beta}{\hbar}\hat{p}\right)$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

Harmonic Oscillator Ground State Wavefunction:

$$\phi_0(x) = \frac{1}{\sqrt{\beta\sqrt{\pi}}}e^{-x^2/2\beta^2}$$

1. **(80 points) Short Answer**

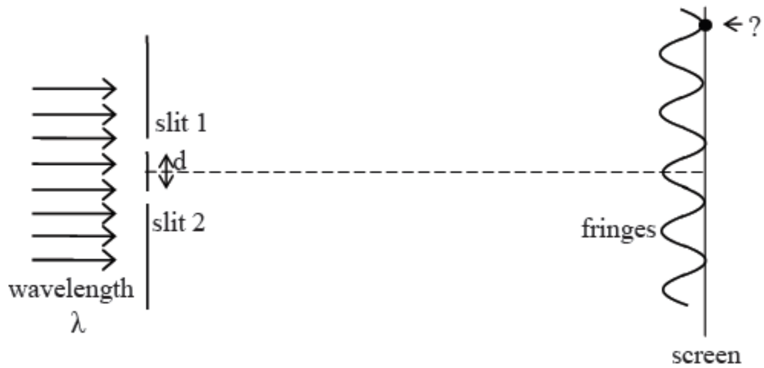
(a) ψ_1 and ψ_2 are momentum eigenfunctions corresponding to different momentum eigenvalues, $p_1 \neq p_2$. Is $\psi = \psi_1 + \psi_2$ also momentum eigenfunction?

Yes No It Depends

(b) A particle of mass m and charge q is accelerated across a potential difference V to a non-relativistic velocity. What is the de Broglie wavelength λ of this particle?

$\frac{m}{\sqrt{2hqV}}$ $\frac{qV}{\sqrt{2mh}}$ $\frac{h}{\sqrt{2mqV}}$ $\frac{m}{\hbar\sqrt{2qV}}$ Something Else

(c) A two-slit interference pattern is viewed on a screen. The position of a particular minimum is marked.



This spot on the screen is further from the lower slit than from the top slit. How much further? Circle one:

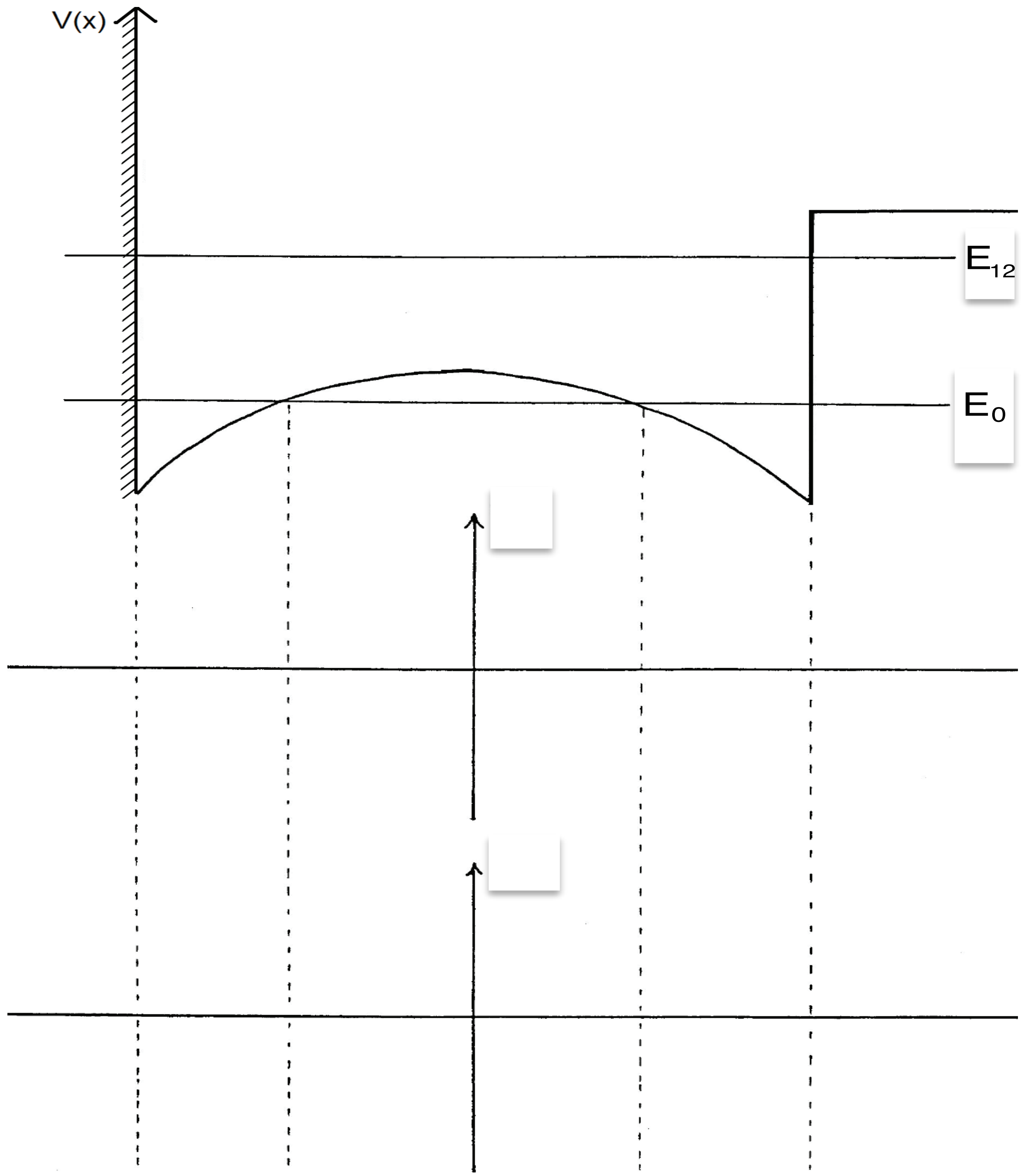
0.5λ 1.5λ 2λ 2.5λ 3λ

(d) Consider a particle of mass m. Is there a physical configuration of the system in which the position in the x direction and the momentum in the x direction can both be predicted with 100% certainty?

Yes, every state Yes, but not all states No, no such state

Yes, but only for free particles Yes, but only for particles in an infinite well

(e) Make qualitative plots of the ground state and the 6^{th} excited state of the potential sketched below, with the lines marked E_0 and E_6 indicating the corresponding energies. Indicate the important features of your sketches.



(f) At $t = 0$, a particle of mass m trapped in an infinite square well of width L is in a superposition of the first excited state and the fifth excited state,

$$\psi_s(x, 0) = A\,(3\phi_1(x) - 2i\phi_5(x)) \ ,$$

where the $\phi_n(x)$ are correctly-normalized energy eigenstates with energies E_n. Which of the following values of A give a properly normalized wavefunction?

$\frac{1}{\sqrt{5}}$ $\qquad$ $\frac{i}{5}$ $\qquad$ $\frac{-i}{\sqrt{13}}$ $\qquad$ $\frac{1}{13}$ $\qquad$ *None of these*

(g) Given the wavefunction ψ_s, what is the probability of measuring the energy to be E_6 at $t = 0$? Circle one:

0 $\qquad$ $\frac{3}{5}$ $\qquad$ $\frac{9}{13}$ $\qquad$ $\frac{9}{25}$ $\qquad$ $\frac{6}{13}$

(h) Given the wavefunction ψ_s, what is the probability density of finding the particle in the middle of the box at time $t = 0$?

0 $\qquad$ $\frac{3}{5}$ $\qquad$ $\frac{9}{13}$ $\qquad$ $\frac{9}{25}$ $\qquad$ *Undetermined*

(i) At time $t = 0$, with the system initially in the state ψ_s, the energy of the system is measured and the largest possible value is found. What is the state of the system immediately after this measurement?

(j) Now suppose that, with the system initially in the state ψ_s, we first measure the position of the particle, and then immediately afterwards we measure the energy of the particle again. What value(s) of the energy could you possibly observe?

(k) MIT scientists have recently discovered a parallel universe in which the laws of physics are completely identical except everyone wears a goatee and/or too much mascara and seems vaguely dangerouss. Your decorated double, who is currently taking the parallel-universe 8.04 exam, just claimed that the wavefunction ψ_s from part (1f) will evolve in time t as,

$$\psi(x,t) = A\,(3\phi_1(x) - 2\,i\phi_5(x))\,e^{iEt}$$

Is your evil twin correct? Circle Yes or No. If Yes, write an incorrect wavefunction in the box below. If No, write the correct wavefunction.

Yes No

(l) Using the correct wavefunction, what is the expectation value $\langle \hat{E} \rangle_t$ at time t in terms of the expectation value $\langle \hat{E} \rangle_0$ at time $t = 0$?

$\langle \hat{E} \rangle_0 e^{-i\omega_1 t}$ $\langle \hat{E} \rangle_0$ $\langle \hat{E} \rangle_0 \cos\left[(\omega_7 - \omega_1)t\right]$ E_1 *None of these*

(m) Let ϕ_n be the properly-normalized n^{th} energy eigenfunction of the harmonic oscillator, and let

$$\psi = \hat{a}\, \hat{a}^\dagger \, \phi_n \, .$$

Which of the following is equal to ψ?

ϕ_n $\quad$ $n\, \phi_{n-1}$ $\quad$ $(n+1)\, \phi_n$ $\quad$ $n\, \phi_{n+1}$ $\quad$ *None of these*

(n) What property of the spectrum of the harmonic oscillator follows from the commutator $[\hat{E}, \hat{a}^\dagger] = \hbar\omega\, \hat{a}^\dagger$? *Note: no computation needed, just a short sentence.*

(o) Consider a harmonic oscillator which is in the state $\psi_*(x,0) = \phi_2$ at time $t = 0$. Will the position probability distribution $\mathbb{P}(x,t)$ vary with time? Circle Yes or No. If yes, write down an specific alternate wavefunction for the harmonic oscillator for which $\mathbb{P}(x)$ is time independent. If no, write one whose $\mathbb{P}(x)$ varies with time.

Yes No

(p) Consider the wavefunction you just identified as having a time-dependent position probability distribution. With what frequency does the position probability distribution oscillate? Construct another wavefunction whose position probability distribution oscillates with twice this frequency.

$frequency:$

(q) Use your knowledge of the operator method to derive the wavefunction for the first excited state of the harmonic oscillator, ϕ_1, from the ground state wavefunction, ϕ_0, given in the formula sheet.

$\phi_1(x) =$ [ ]

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2. **(20 points) Particle in Mystery Potential**

The wavefunction for a particle of mass m moving in a potential $V(x)$ is given by

$$\psi(x,t) = \left\{ \begin{array}{cc} x\, e^{-Bx}\, e^{-iCt/\hbar} & x > 0 \\ 0 & x < 0 \end{array} \right\}$$

where B and C are real constants such that $\psi(x,t)$ is a properly normalized wave function that obeys the Schrödinger time-evolution equation for a potential $V(x)$.

(a) Sketch this wavefunction at time $t = 0$. Mark any significant features.

(b) Using what you know about ψ, make a qualitative sketch of the potential $V(x)$ governing this system, indicating in particular any classically forbidden regions and classical turning points.

(c) Is this particle in a state corresponding to a definite energy? If so, what is the energy (in terms of any or all of B and C); if not, why not?

Yes No

(d) Are there any energy eigenstates in this potential with lower energy than ψ? Explain (briefly).

Yes No

(e) (5 Point Bonus) Determine the potential $V(x)$ in terms of B, C, m, and $\hbar$. Does your result agree with your qualitative sketch?

$V(x) =$ [ ]

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8.04 Quantum Physics I
Spring 2013