Problem Set 3

Due **Tuesday Feb 26** *at 11.00AM*

Assigned Reading:

E&R	5.(all)	
Li.	3.(1-3)	5.(2-4)
Ga.	2.(all)	3.2
Sh.	1, 4	

1. **(20 points) Mathematical Preliminaries: Linear Operators**

An operator $\hat{\mathcal{O}}$ is said to be *Linear* if, for any functions $f(x)$ and $g(x)$ and for any constants a and b,

$$\hat{\mathcal{O}}\,(a f(x) + b\, g(x)) = a\,\hat{\mathcal{O}} f(x) + b\,\hat{\mathcal{O}} g(x)$$

(a) Which of the following operators are linear?

$$\begin{array}{ll}
\hat{\mathbb{1}} = \text{identity} & \hat{\mathbb{1}}\, f(x) = f(x) \\
\hat{S} = \text{squares } f(x) & \hat{S}\, f(x) = f^2(x) \\
\hat{D} = \frac{\partial}{\partial x} & \hat{D}\, f(x) = \frac{\partial f(x)}{\partial x} \\
\hat{I}_x = \int_0^x dx' & \hat{I}_x\, f(x) = \int_0^x dx' f(x') \\
\hat{A} = \text{adds } 3 & \hat{A}\, f(x) = f(x) + 3 \\
\hat{\mathbb{P}}_g = \text{maps to } g(x) & \hat{\mathbb{P}}_g\, f(x) = g(x) \\
\hat{T}_L = \text{translates by L} & \hat{T}_L\, f(x) = f(x - L)
\end{array}$$

(b) What are the eigenfunctions and eigenvalues of the following operators:

$$\begin{aligned}
\hat{\mathbb{1}}\, f(x) &= f(x) \\
\hat{S}\, f(x) &= f^2(x) \\
\hat{x}\, f(x) &= x\, f(x) \\
\hat{D}\, f(x) &= \partial_x f(x)
\end{aligned}$$

Explain how your answers change if the space of functions on which your operator acts is (i) arbitrary functions on the real line (ii) continuous functions on the real line (iii) continuous functions on the real line which do not diverge at infinity (iii) square-normalizable continuous functions on the real line.

(c) Show that the eigenfunctions of $\hat{T}_L$ are of the form $\phi_\beta(x) = e^{\beta x} g(x)$, where β is a complex number and g is periodic with period L, $g(x+L) = g(x)$. What is the eigenvalue of $\hat{T}_L$ when acting on ϕ_β?

2. **(25 points) Mathematical Preliminaries: More on the Translation Operator**
As we've seen, the "position" operator, $\hat{x}$, acts on functions of position, $f(x)$, as,

$$\hat{x}\, f(x) = x\, f(x)\,.$$

Meanwhile, the "derivative" operator, $\hat{D}$, acts on functions of position, $f(x)$, as,

$$\hat{D}\, f(x) = \partial_x f(x)\,,$$

while the "translate–by–L" operator, $\hat{T}_L$, acts on functions of position, $f(x)$, as,

$$\hat{T}_L\, f(x) = f(x-L)\,.$$

(a) Show that
$$[\hat{T}_L, \hat{x}] = -L\,\hat{T}_L.$$
Note: Two operators are equal if both operators act identically on an arbitrary function $f(x)$. In equations, $\hat{A} = \hat{B}$ if $\hat{A}f(x) = \hat{B}f(x)$ for any function $f(x)$.

(b) Show that $\hat{T}_L$ commutes with the derivative operator, *i.e.* that
$$[\hat{T}_L, \hat{D}] = 0.$$

(c) Show that
$$\hat{T}_L = e^{-L\hat{D}}$$
Hint: To define the RHS, use the Taylor expansion $e^x = 1 + x + \frac{1}{2}x^2 + ...$

(d) Use (a) and (c) to show that
$$\left[\hat{D}, \hat{x}\right] = \hat{\mathbb{1}}.$$

(e) If $\hat{T}_L\, f(x) = f(x-L)$, how does $\hat{T}_L$ act on $\tilde{f}(k)$, the fourier transform of $f(x)$? In other words, what modification of $\tilde{f}(k)$ corresponds to translating $f(x)$ by L?

(f) Use parts (c) and (e) to determine how $\hat{D}$ acts on $\tilde{f}(k)$. Does this make sense?

(g) Use to the definition of the position operator and the definition of the fourier transform to determine how $\hat{x}$ acts on the fourier transform, $\tilde{f}(k)$, of $f(x)$.

(h) Verify that the commutation relation $\left[\hat{D}, \hat{x}\right] = \hat{\mathbb{1}}$ holds whether acting on a function $f(x)$ or its fourier transform, $\tilde{f}(x)$. What lesson do you take from this?

3. **(25 points) Mathematical Preliminaries: The Meaning of the Commutator**
Let $\hat{A}$ and $\hat{B}$ be linear operators, and let $\hat{C}$ denote their commutator, *i.e.*

$$\hat{C} \equiv [\hat{A}, \hat{B}]\,.$$

(a) Show that $\hat{C}$ is also a linear operator.

(b) Suppose $\hat{A}$ and $\hat{B}$ share a common eigenfunction, ϕ_{ab}, *i.e.*

$$\hat{A}\phi_{ab} = a\phi_{ab} \quad \text{and} \quad \hat{B}\phi_{ab} = b\phi_{ab}\,.$$

Show that ϕ_{ab} must be annihilated by the commutator, *i.e.*

$$\hat{C}\phi_{ab} = 0\,.$$

(c) Suppose $[\hat{A}, \hat{B}] = 0$. Such operators are said to *commute*. Can $\hat{A}$ and $\hat{B}$ share common eigenfunctions?

(d) Suppose $\hat{A}$ and $\hat{B}$ commute. Are all eigenfunctions of $\hat{A}$ necessarily also eigenfunctions of $\hat{B}$? If so, explain why; if not, give a simple counterexample.

4. **(30 points) Operators and Commutators in Quantum Mechanics**

As we have seen, in quantum mechanics, position and momentum are represented by linear operators acting on the wavefunction, $\psi(x)$, as

$$\hat{p}\,\psi(x) = \frac{\hbar}{i}\partial_x\psi(x) \qquad \hat{x}\,\psi(x) = x\,\psi(x)\,.$$

Representing observables as operators implies a host of facts about observables in quantum mechanics. In this problem we will use the results you derived above about mathematical operators to deduce properties of observables in quantum mechanics.

(a) Using your results in the previous three problems, together with the representations of the momentum and position given above, show that the position-momentum commutator takes the value,

$$[\hat{p}, \hat{x}] = \frac{\hbar}{i}\hat{\mathbb{1}}$$

(b) Can $\hat{x}$ and $\hat{p}$ share any common eigenfunctions? If so, give an example; if not, explain why. Physically, what does this mean?

(c) Can $\hat{p}$ and $\hat{T}_L$ share any common eigenfunctions? If so, give an example; if not, explain why. Physically, what does this mean?

(d) Suppose $\hat{A}$ and $\hat{B}$ are the Quantum operators representing two observables of a physical system. What must be true of the commutator $[\hat{A}, \hat{B}]$ so that the system can have definite values of A and B simultaneously?

(e) Discuss the connection between the uncertainty principle and commutation relations amongst the operators representing observables in Quantum Mechanics. (Think about this deeply, then discuss briefly. We will return to this in detail later in the semester.)

(f) Let $\hat{H}$ denote the operator representing the "hardness" of an electron, and $\hat{C}$ the "color". Do $\hat{H}$ and $\hat{C}$ commute? Explain.

(g) Let ϕ_H denote the eigenfunction of $\hat{H}$ with eigenvalue $+1$ ("hard"), and ϕ_S that with eigenvalue -1 ("soft"). Similarly, let ϕ_B and ϕ_W be the eigenfunctions of $\hat{C}$ with eigenvalues $+1$ and -1 ("black" and "white"). In the first lecture we argued that a white electron is a superposition of hard and soft; ditto a black electron. Using everything you've learned thus far, propose explicit representations of ϕ_W and ϕ_B as superpositions of ϕ_H and ϕ_S which are compatible with all of the data presented in the first lecture.

(h) Why are linear operators so important in quantum mechanics? (Think about this deeply, then discuss briefly.)