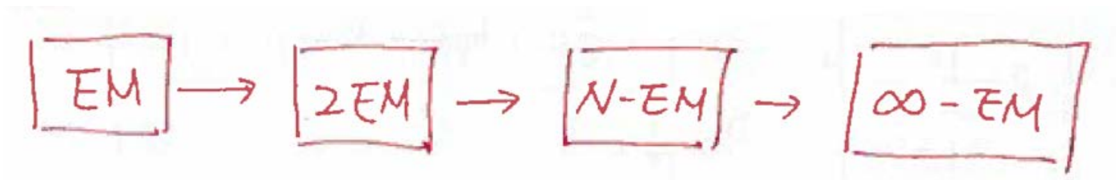
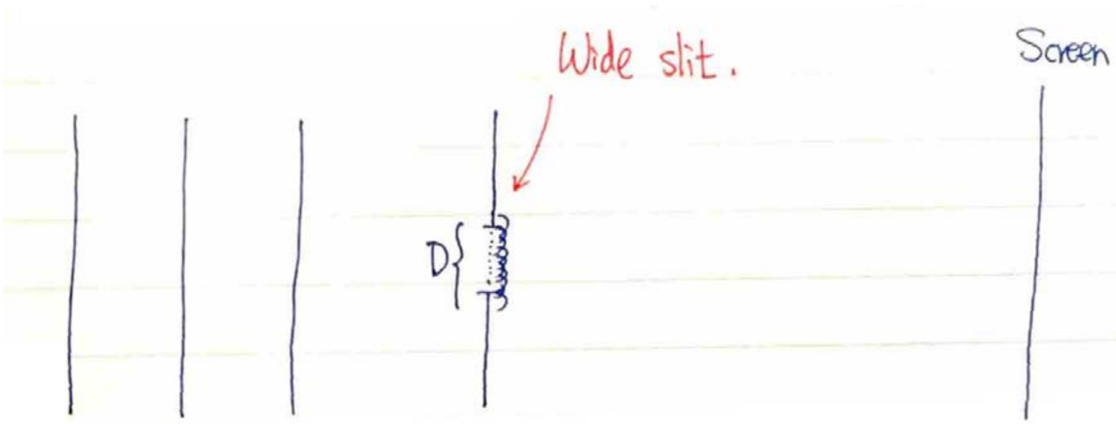
8.03 Lecture 22

We learned the interference of two EM waves to N EM waves.

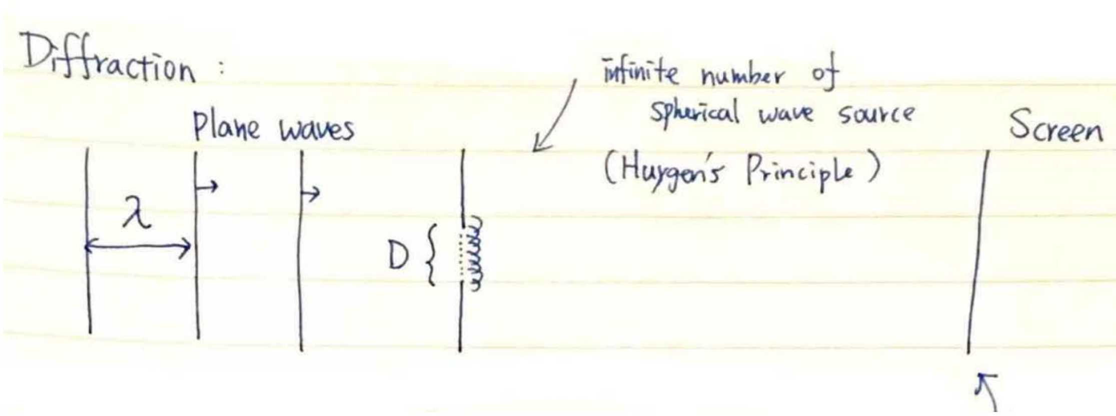


We call the interference of infinite number of EM waves "diffraction".



We have ∞ point like spherical EM wave sources. This situation: we will see the "interference" between all the spherical wave sources. We call it "diffraction".

Feynman: No one has ever been able to define the difference between interference and diffraction satisfactorily. It is just a question of usage.



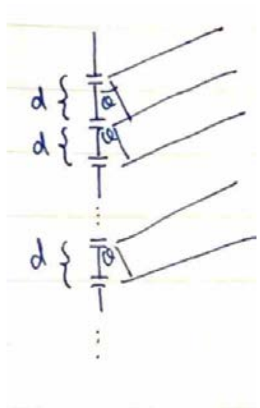
What is the resulting intensity pattern?

⟨ Method I ⟩

Reminder: N-slit interference:

$$\langle I \rangle \propto \left[\frac{\sin\left(\frac{N\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)} \right]^2$$

Where δ is the phase difference between near-by slits: $\delta = \frac{d \sin\theta}{\lambda} 2\pi$



Consider the limit:

$$d \longrightarrow 0 \qquad N \longrightarrow \infty \qquad Nd = D$$

$$\Rightarrow \delta \longrightarrow 0 \qquad N\delta = \frac{D \sin\theta}{\lambda} 2\pi$$

$$\langle I \rangle \propto \left[\frac{\sin\left(\frac{N\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)} \right]^2$$

We can define:

$$\beta = \frac{N\delta}{2} = \frac{\pi D \sin\theta}{\lambda}$$

$$\Rightarrow \langle I \rangle \propto \left[\frac{\sin\beta}{\beta} \right]^2$$

Here we also assume that the intensity of individual point source is proportional to N^{-2}.

⟨ Method II ⟩

Another method described in Georgi's book: Do an integration over all point-like sources to calculate the total electric field

$$C(k_x, k_y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy f(x,y) e^{-i\vec{k}\cdot\vec{r}(x,y)}$$

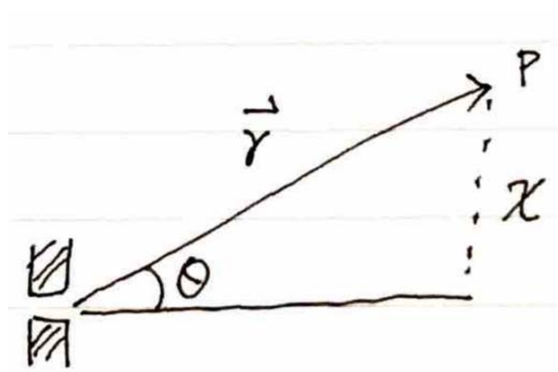
Where C is proportional to the total electric field. The integrals are over the unite area of the point source and f is the shape of the sources. This is the Fourier transform of $f(x, y)$

Let's consider a single slit experiment

$$f(x,y) = \begin{cases} 1 \text{ if } \frac{-D}{2} \le x \le \frac{D}{2} \\ 0 \text{ if } |x| > \frac{D}{2} \end{cases}$$

$$\begin{aligned} C(k_x, k_y) &= \frac{1}{4\pi^2}\int_{-D/2}^{D/2} e^{-ik_x x}\, dx \int_{-\infty}^{\infty} e^{-ik_y y}\, dy \\ &= \delta(k_y)\frac{1}{2\pi}\frac{1}{-ik_x} e^{-ik_x x}\Big|_{-D/2}^{D/2} \\ &= \delta(k_y)\frac{1}{2\pi}\frac{1}{-ik_x}\left[e^{-ikD/2} - e^{+ikD/2}\right] \\ &= \delta(k_y)\frac{1}{2\pi}\frac{2\sin k_x D/2}{k_x} \end{aligned}$$

Therefore



$$\begin{aligned} |\vec{E}| &\propto C \propto \frac{\sin k_x D/2}{k_x} \\ I &\propto |C|^2 \propto \frac{\sin^2 k_x D/2}{k_x} \\ \text{since} \quad \frac{x}{r} &= \frac{k_x}{k} = \frac{k_x \lambda}{2\pi} = \sin\theta \end{aligned}$$

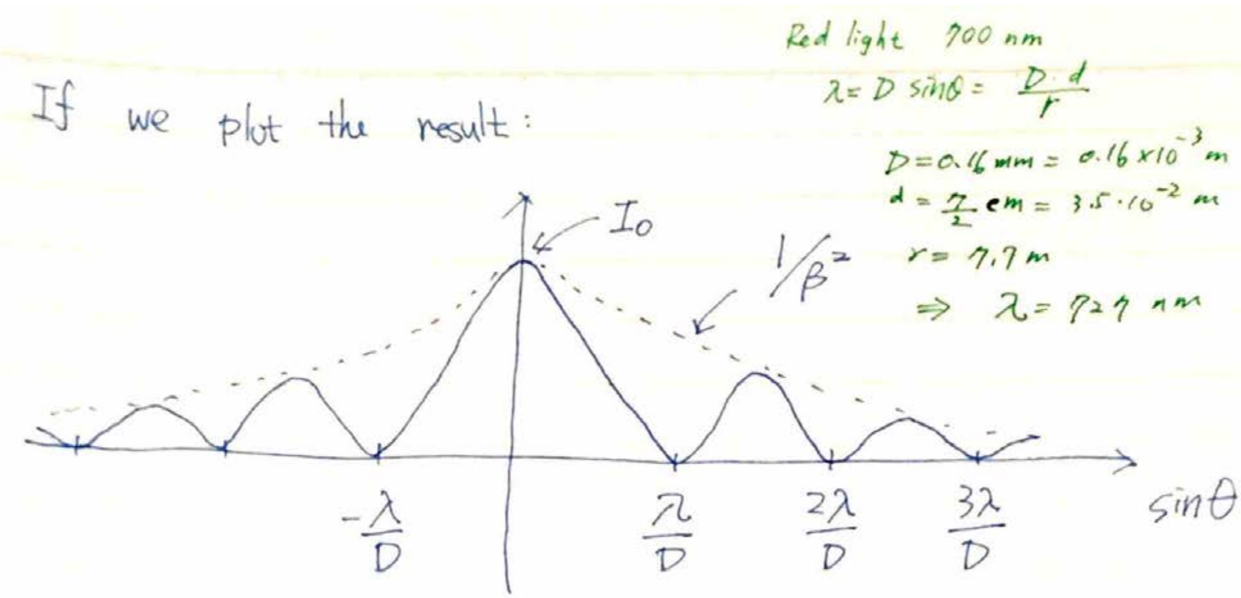
$$\Rightarrow k_x = \frac{2\pi \sin\theta}{\lambda}$$

$$\Rightarrow I \propto \frac{\sin^2\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\left(\frac{\pi D}{\lambda}\sin\theta\right)^2}$$

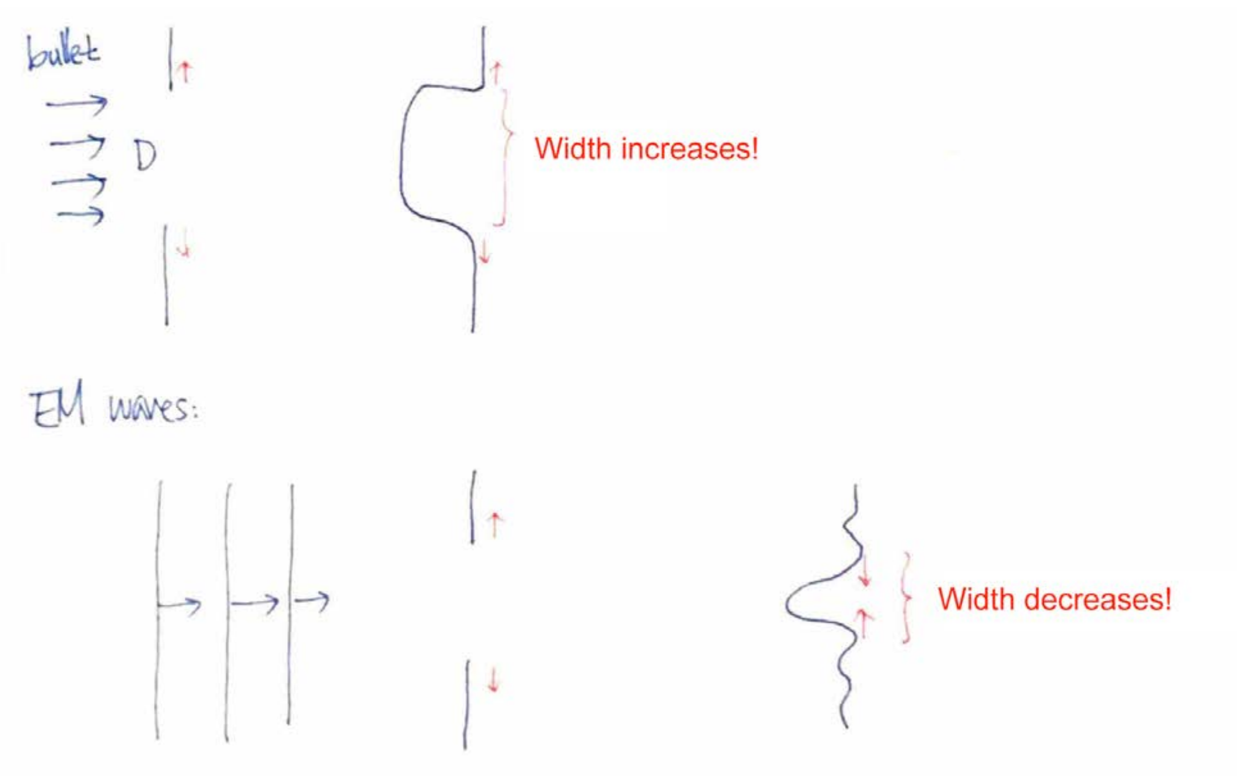
$$\text{Define } \beta \equiv \frac{\pi D \sin\theta}{\lambda}$$

$$\langle I \rangle \propto \left[\frac{\sin\beta}{\beta}\right]^2$$

Same result as method I!

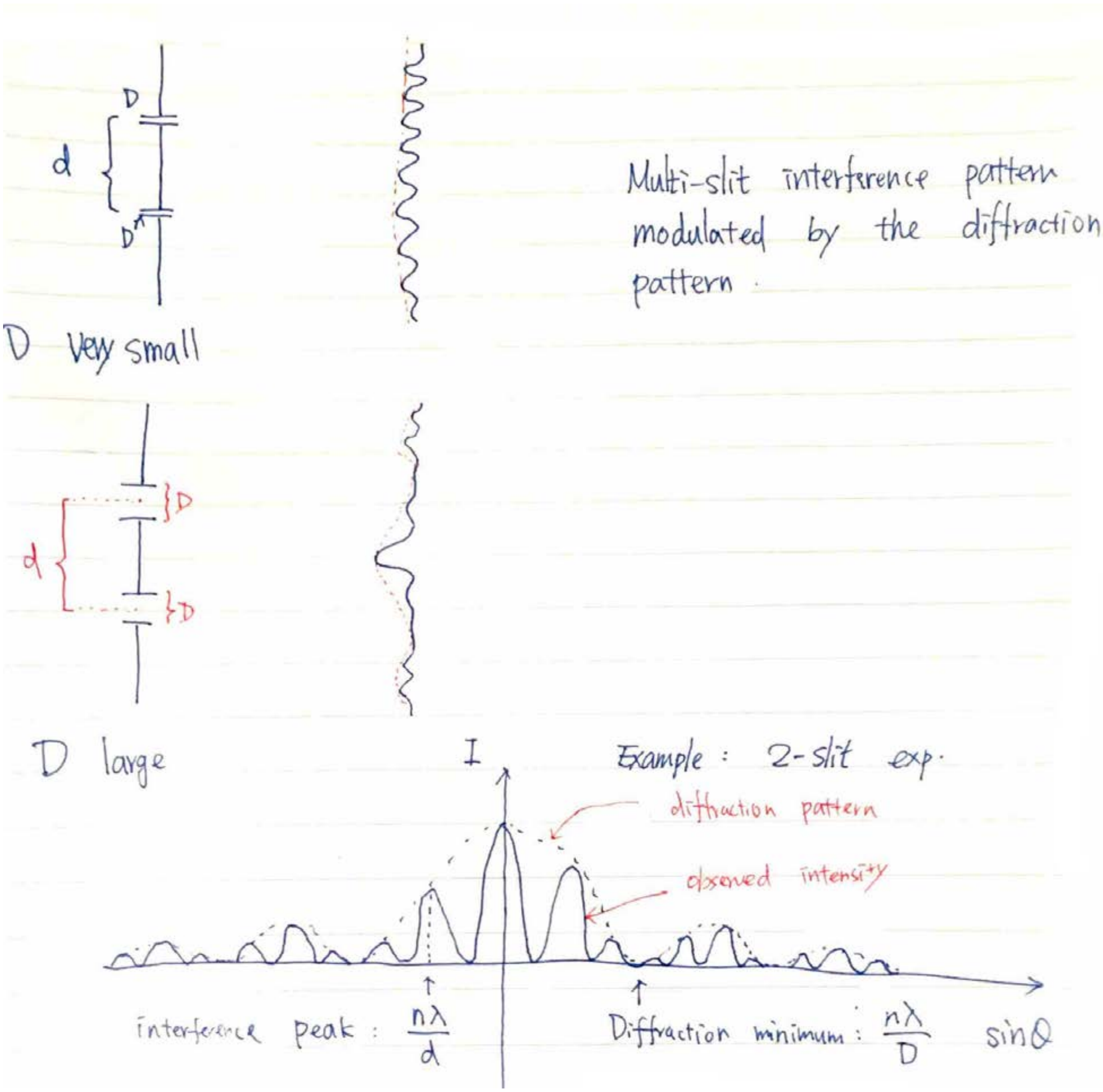


Observation:
(1) If we increase the size of the slit D:
⇒ the width decreases!



(2) Distance between peaks $\propto \lambda$
Principle Maximum $\Rightarrow$ The width is larger for red light (longer wave length) than blue light (shorter wave length).
(3) Intensity decreases quickly $\propto \dfrac{1}{\beta^2}$ as a function of β (or $\sin\theta$) if D is large . On the other hand: if D is smaller, intensity decreases slower.

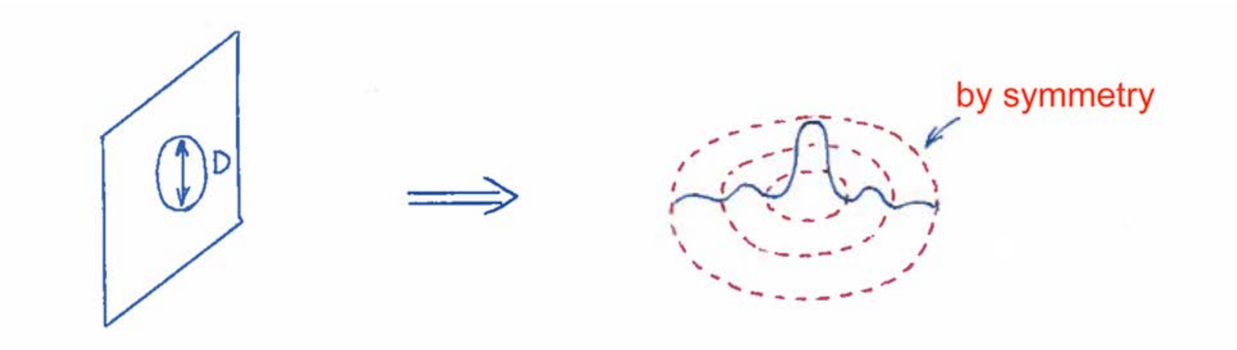
Coming back to the double-slit experiment: make it even more realistic: include the effect from finite slit width:



$$I = I_0\Big(\underbrace{\frac{\sin\beta}{\beta}}_{\text{Diffraction}}\Big)^2\Big(\underbrace{\frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}}_{\text{Multi-slit interference}}\Big)^2$$

$$\beta = \frac{\pi D}{\lambda}\sin\theta \qquad \delta = kd\sin\theta = \frac{2\pi d\sin\theta}{\lambda}$$

Let's consider a pin hole or aperture:



One can do the integration and we found that the intensity is:

$$I(\theta) = I_0\Big(\frac{J_1(\beta)}{\beta}\Big)^2$$

Where J_1 is a Bessel function of the first kind:
Solve:

$$J_1(x) = 0 \quad \Rightarrow \quad x \approx 3.83$$

$$\Rightarrow \beta = 3.83 = \frac{\pi D}{\lambda}\sin\theta$$

$$\Rightarrow \sin\theta \approx 1.22\frac{\lambda}{D}$$

And so the resolution of a pin hole:

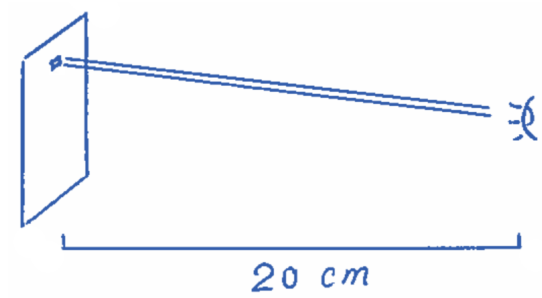
$$\sin\Delta\theta \approx \Delta\theta = 1.22\frac{\lambda}{D}$$

Such that we can separate the two peaks! Human pupil is 2-4 mm when narrow and 3-8 mm when wide. Take visible light which is around 500 nm. $D \sim 5$ mm. Resolution:

$$\sim 1.22\frac{\lambda}{D} \sim 1.22\frac{5\cdot 10^{-7}}{5\cdot 10^{-3}} \sim 1.22\cdot 10^{-4}$$

iPhone 7: 401 ppi

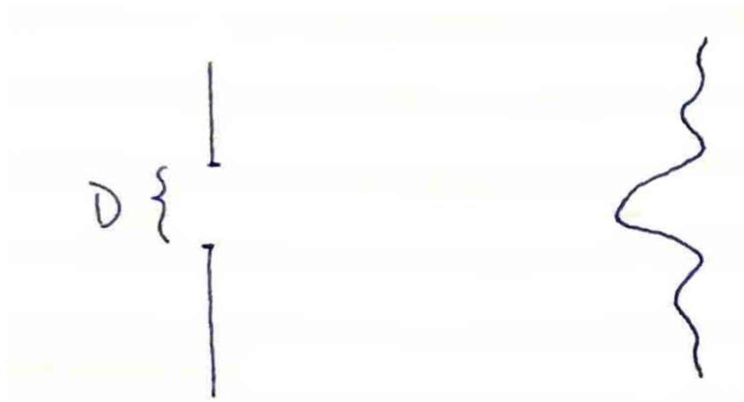
$$\Delta x \sim \frac{2.54\text{cm}}{401} \sim 6.3 \times 10^{-3} \text{ cm}$$
$$\Delta\theta \sim \frac{\Delta x}{10 \text{ cm}} \sim 3 \times 10^{-4}$$



The human eye can resolve it! Will you buy the iPhone x with 40,000 ppi? If Apple put 2,000 pixels in 6 cm $\sim$ the limit.

We have learned single slit diffraction.

$$I = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \qquad \beta = \frac{\pi D \sin\theta}{\lambda}$$



This means that a laser pointer is not merely producing a pencil beam.

Suppose $\lambda = 500$ nm $= 5 \times 10^{-7}$ m and $D = 1$ mm $= 1 \times 10^{-3}$ m.

Opening angle:

$$\theta \approx 1.22\frac{\lambda}{D} = 6 \times 10^{-4}$$

If we shoot a laser to moon: $L = 4 \times 10^8$ m the radius of the principle maxima is 240 km!!