Review:

* Fourier Transform
* Narrow band signal transmission.

$$f(t) = f_s(t) \cos \omega_0 t$$

$$\Big\downarrow \text{ small bandwidth approximation}$$

$$\psi(x,t) \approx \mathrm{Re}\left(\underbrace{f_s\left(t-\frac{x}{v_g}\right)} \underbrace{e^{-i\left(t-\frac{x}{v_p}\right)\omega_0}} \right)$$

The envelope traveling at $v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$ — Group velocity!

The carrier traveling at $v_p = \frac{\omega_0}{k_0}$ — Phase velocity!!

AM radio:

Typical ω_0 : 0.3 − 30MHz

$\Delta\omega \ll \omega_0$

$\Delta\omega$: 5kHz

Make sense!

Example :

$$f(t) = e^{-\Gamma|t|}$$

What will be the corresponding $C(\omega)$?

From previous lecture :

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \; e^{-\Gamma|t|} \, e^{i\omega t}$$

$$= \frac{1}{2\pi}\left[\underbrace{\int_{-\infty}^{0} dt \; e^{+\Gamma t} \, e^{i\omega t}}_{= \frac{1}{\Gamma + i\omega}} + \underbrace{\int_{0}^{\infty} dt \; e^{-\Gamma t} \, e^{i\omega t}}_{= \frac{-1}{-\Gamma + i\omega}} \right]$$

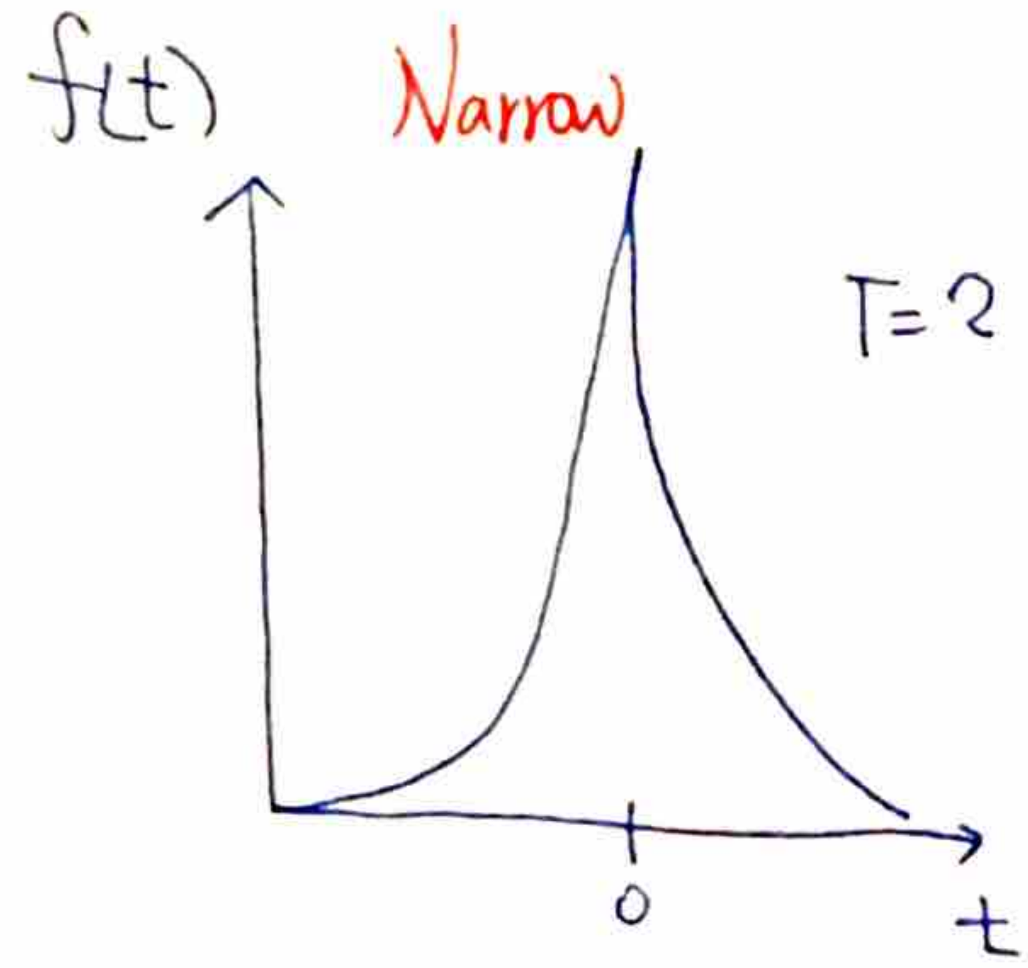
$$= \frac{1}{2\pi}\left[\frac{1}{\Gamma + i\omega} + \frac{1}{\Gamma - i\omega} \right]$$

$$= \frac{1}{2\pi} \frac{2\Gamma}{\Gamma^2 + \omega^2}$$

$$= \frac{\Gamma}{\pi(\Gamma^2 + \omega^2)}$$

If we plot $C(\omega)$ as a function of ω

Georgi P.235



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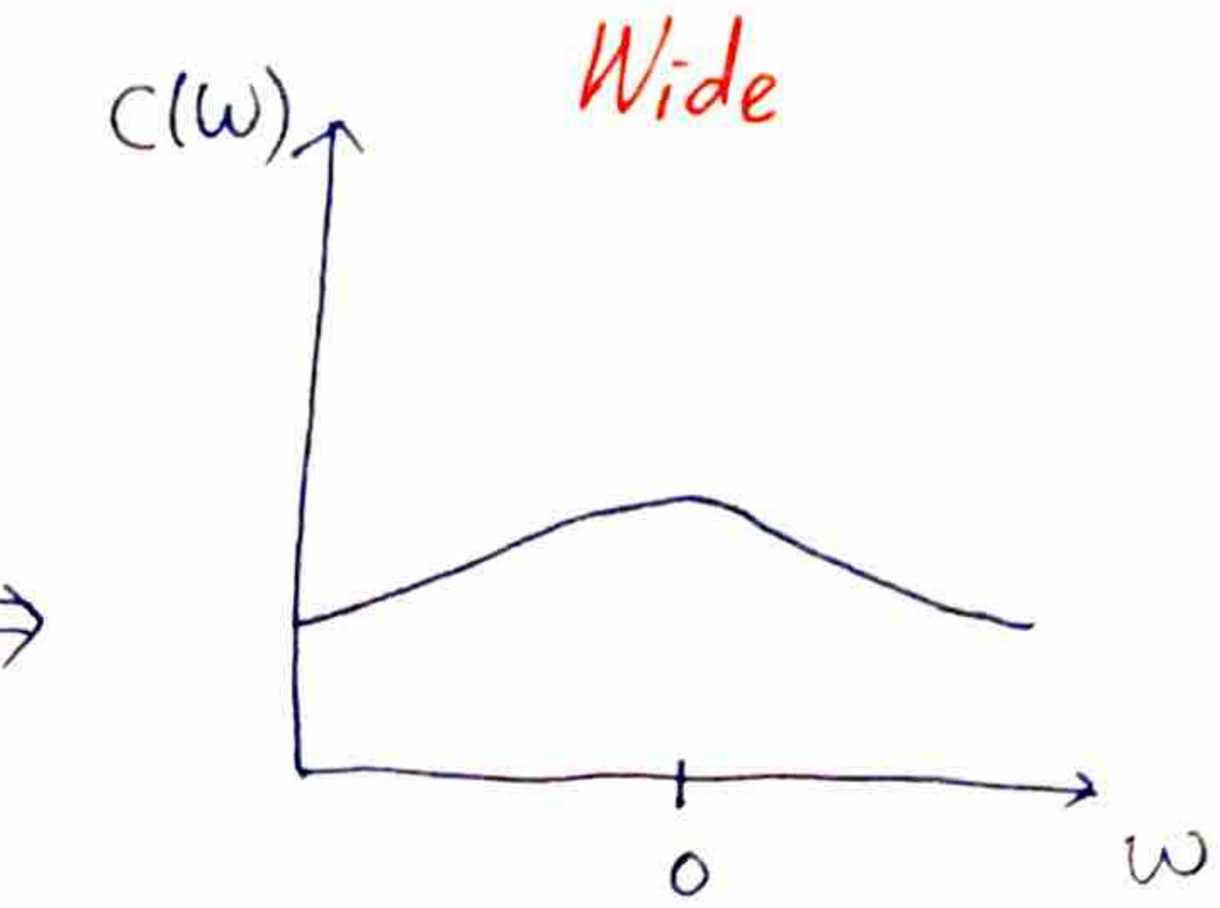

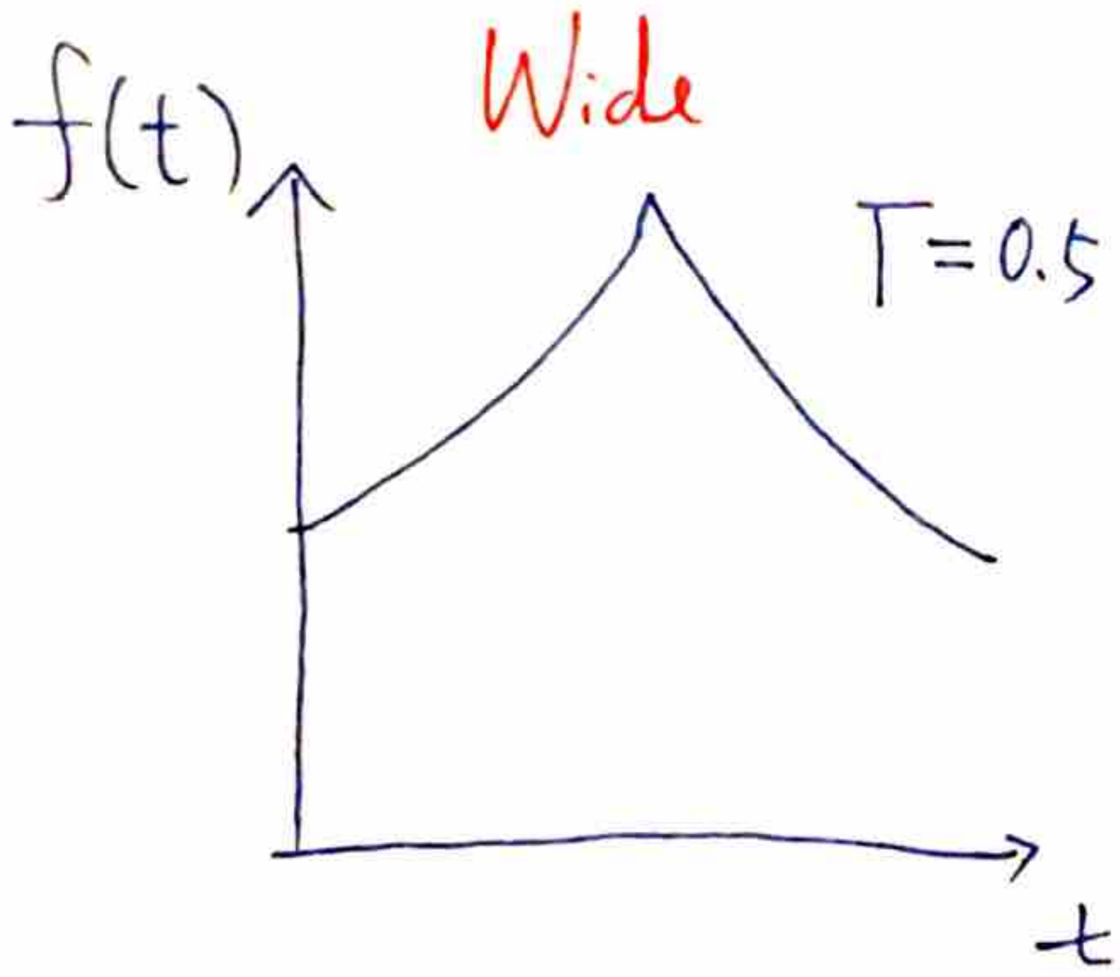

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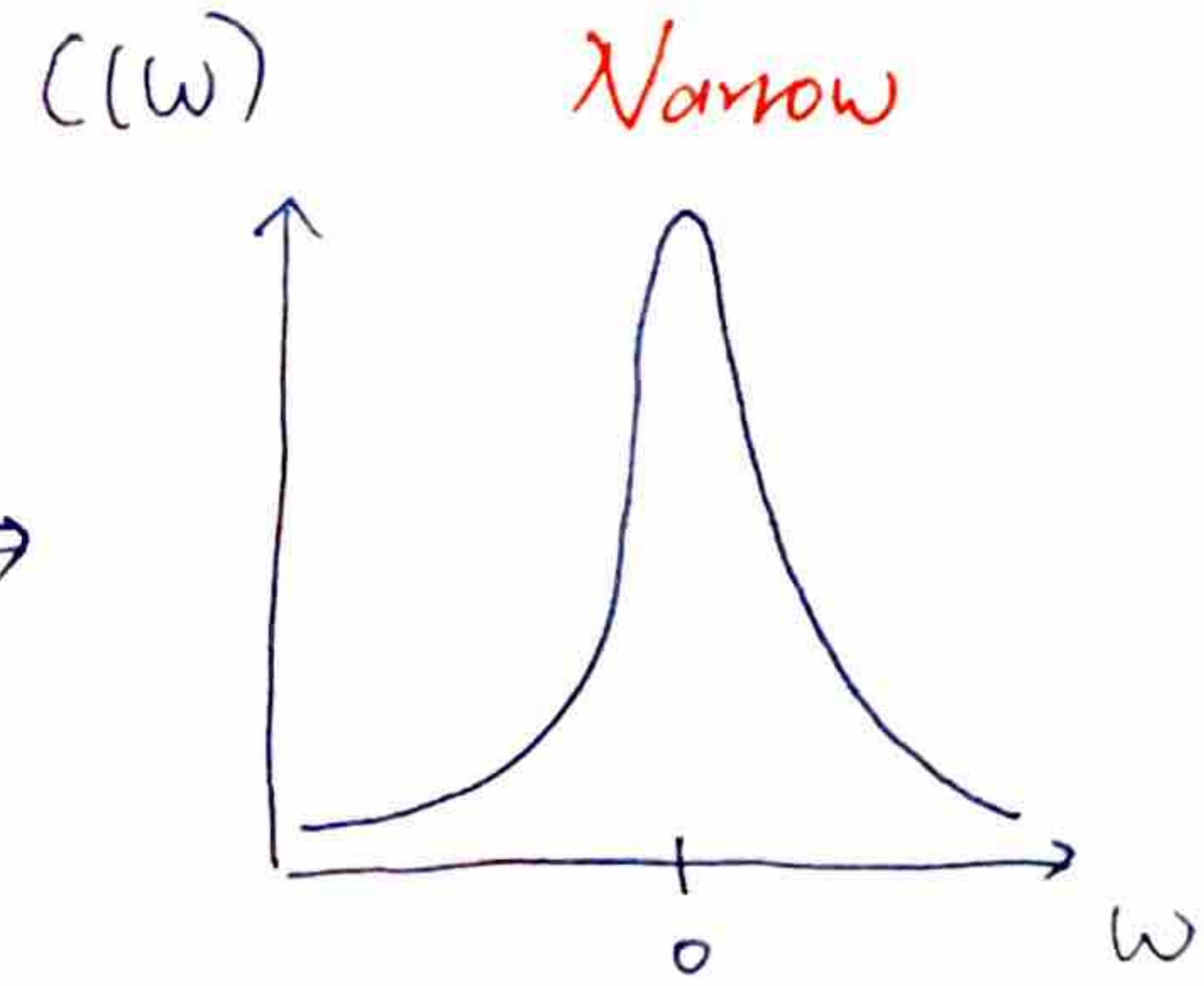


Large T ⇒ Narrow Pulse ($f(t)$ narrow)

but one will get wide $C(\omega)$

Small T ⇒ Wide Pulse ($f(t)$ wide) Spread out

the one will get narrow $C(\omega)$!

In your pset, you will work on another functional form: Gaussian Wave.

We can demonstrate this using a precise mathematical definition of the spread of the signal.

(1) We define the intensity of the signal to be proportional to $|f(t)|^2$

(2) Average value of any function weighted with the signal's intensity:

$$\langle g(t) \rangle = \frac{\int_{-\infty}^{\infty} dt\ g(t)\, |f(t)|^2}{\int_{-\infty}^{\infty} |f(t)|^2}$$

(3) Spread of time:

$$\Delta t^2 \equiv \langle [t - \langle t \rangle]^2 \rangle = \frac{\int_{-\infty}^{\infty} dt\ |(t-\langle t \rangle) f(t)|^2}{\int_{-\infty}^{\infty} |f(t)|^2}$$

mean square deviation from the average time.

Similarly spread of the frequency spectrum:

$$\Delta \omega^2 \equiv \langle [\omega - \langle \omega \rangle]^2 \rangle$$

We will prove: $\Delta \omega \cdot \Delta t \geq \frac{1}{2}$

(4) We also realize that

$$\int_{-\infty}^{\infty} d\omega \; \omega \; c(\omega) e^{-i\omega t} = i\frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\omega \; c(\omega)\, e^{-i\omega t}$$

$$= i\frac{\partial}{\partial t} f(t)$$

$$\Rightarrow \langle \omega \rangle = \frac{\int_{-\infty}^{\infty} dt \; f(t)^* \, i\frac{\partial}{\partial t} f(t)}{\int_{-\infty}^{\infty} dt \, |f(t)|^2}$$

$$\Delta\omega^2 \equiv \langle [\omega - \langle\omega\rangle]^2 \rangle = \frac{\int_{-\infty}^{\infty} dt \left| \left(\overset{\omega}{i\frac{\partial}{\partial t}} - \langle \omega \rangle\right) f(t) \right|^2}{\int_{-\infty}^{\infty} dt \, |f(t)|^2}$$

Take home message: $\omega \leftrightarrow i\frac{\partial}{\partial t}$

(5) We will use a trick which leads to the Heisenberg uncertainty principle in Quantum Mechanics.

Consider a function $\gamma(t)$

$$\gamma(\kappa, t) \equiv \left([t - \langle t \rangle] - i\kappa \left[i\frac{\partial}{\partial t} - \langle \omega \rangle \right] \right) f(t)$$

κ : a free parameter

$$\equiv \left(T - i\kappa\, \Omega \right) f(t)$$

Consider
$$R(\kappa) = \frac{\int_{-\infty}^{\infty} dt \, |\gamma(\kappa,t)|^2}{\int_{-\infty}^{\infty} dt \, |f(t)|^2}$$

$\because$ Both numerator and denominator are positive

$\Rightarrow \quad R > 0$

$$|\gamma(\kappa,t)|^2 = (T - i\kappa\Omega) f \cdot (T + i\kappa\Omega^*) f^*$$
$$= \underset{①}{|Tf|^2} + \underset{②}{|\Omega f|^2} + \underset{③}{i\kappa \left[Tf \underline{\Omega^*} f^* - \underline{\Omega} f T f^* \right]}$$

$$③ : i\kappa \left[Tf \underline{\left(-i \tfrac{\partial}{\partial t} - \cancel{\langle \omega \rangle}\right)} f^* - \underline{\left(i \tfrac{\partial}{\partial t} - \cancel{\langle \omega \rangle}\right)} f \, Tf^* \right]$$
$$= \kappa T \left[f \frac{\partial f^*}{\partial t} + \frac{\partial f}{\partial t} f^* \right]$$
$$= \kappa T \frac{\partial}{\partial t} (f f^*)$$

$$\int_{-\infty}^{\infty} dt \, ③ = \int_{-\infty}^{\infty} dt \, \kappa T \frac{\partial}{\partial t}(f f^*) = \underbrace{\kappa T f f^* \Big|_{-\infty}^{\infty}}_{=0} - \kappa \int_{-\infty}^{\infty} |f|^2 \underbrace{\frac{\partial T}{\partial t}}_{=1} dt$$

$$= -\kappa \int_{-\infty}^{\infty} dt \, |f|^2$$

Assuming f is localized
$f(\pm\infty) = 0$

$\Rightarrow$

① : $$\frac{\int_{-\infty}^{\infty} [t-\langle t\rangle]^2 |f(t)|^2 dt}{\int_{-\infty}^{\infty} dt\, |f(t)|^2} = \Delta t^2$$

② : $$\frac{K^2 \int_{-\infty}^{\infty} \left|\left[i\frac{\partial}{\partial t} - \langle \omega\rangle\right] f(t)\right|^2}{\int_{-\infty}^{\infty} dt\, |f(t)|^2} = K^2 \Delta\omega^2$$

$$\Rightarrow ③ \rightarrow \frac{K \int_{-\infty}^{\infty} dt\, |f|^2}{\int_{-\infty}^{\infty} dt\, |f|^2} = -K$$

$$\Rightarrow R = \Delta t^2 + K^2 \Delta\omega^2 - K > 0$$

$$\left(\frac{dR}{dK} = 0\right)$$

$R(K)$ minimize at $K = K_{min} = \dfrac{1}{2\Delta\omega^2}$

$$\Rightarrow R(K_{min}) = \Delta t^2 - \frac{1}{4\Delta\omega^2} \geq 0$$

$$\Rightarrow \Delta t \cdot \Delta\omega \geq \frac{1}{2}$$

Uncertainty Principle !!!!

AM station broadcast :

$$\text{Band width} : \quad \Delta\omega = 2\pi \underbrace{\Delta\nu}_{\sim 5\,KHz} \approx 3\times10^{4}\, s^{-1}$$

$\Rightarrow$ Δt is a few $\times 10^{-5}$ s.

$\Rightarrow$ We can not tell signals with $\Delta t < O(10^{-5} s)$

ok.

Not ok, not separable.

Quantum Mechanics :

We can rewrite it :

$$\nu \Delta t \cdot \frac{\Delta\omega}{\nu} \geq \frac{1}{2}$$

$$\Rightarrow \quad \Delta x \cdot \Delta k \geq \frac{1}{2}$$

In quantum mechanics $\quad p = \hbar k$

Planck constant

$\hbar = \frac{h}{2\pi}$

$\hbar \sim 6.6 \times 10^{-16}$ eV·s

$\sim 1 \times 10^{-34}$ J·s

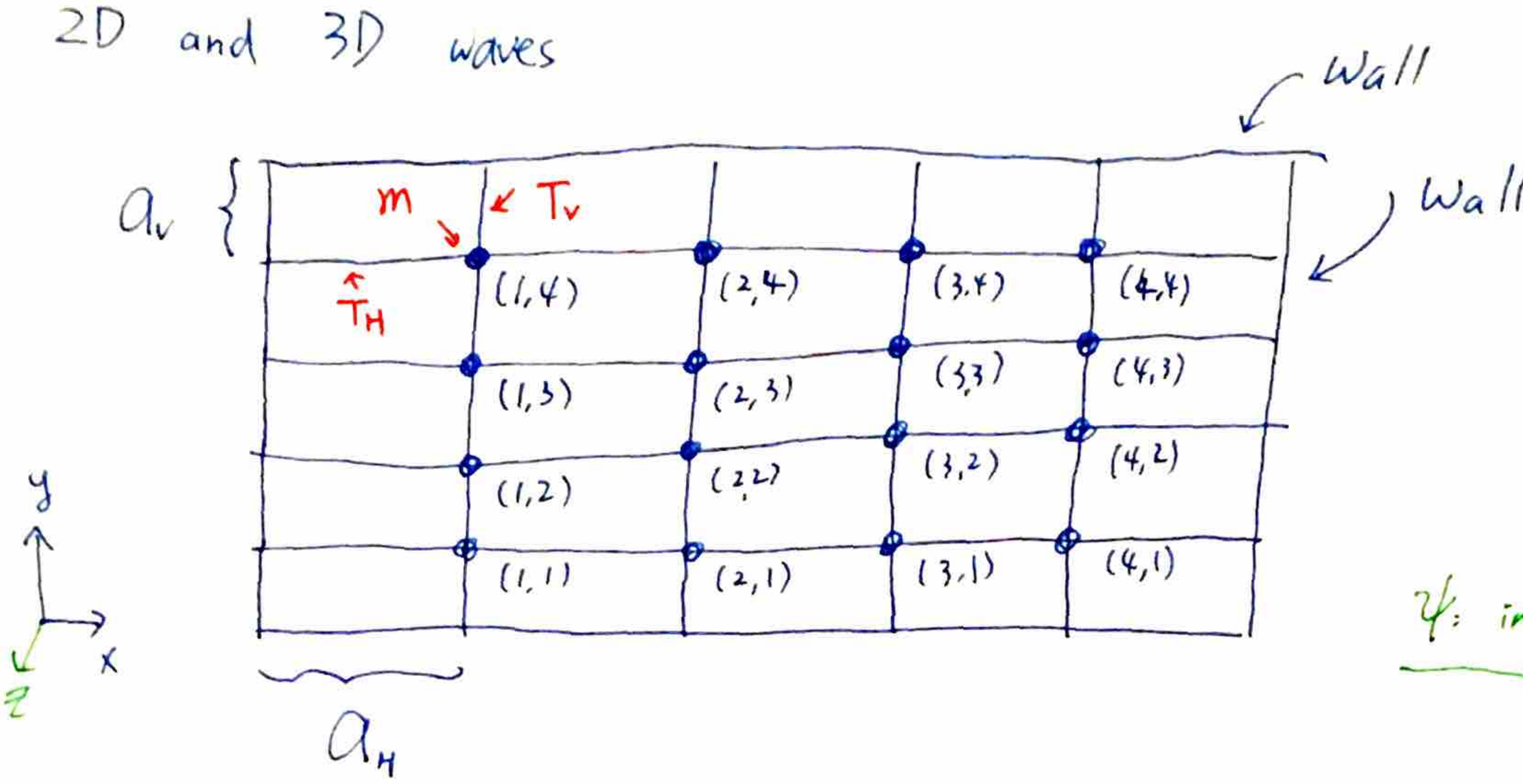
$$\Rightarrow \quad \Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

This is the mathematical statement :

The position and momentum of a particle cannot be specified simultaneously.

Speed of light. slides Break?

2D and 3D waves



ψ: in Z direction

In this example: two-dimentional beaded mesh

Index: (j_x, j_y) in this case: j_x: $1 \sim 4$

j_y: $1 \sim 4$

First consider infinite system by removing the wall

$\Rightarrow$ Space translation symmetry:

X direction: Eigenstate of $M^{-1}K$ is $e^{ik_x x}$

Y direction: Eigenstate of $M^{-1}K$ is $e^{ik_y y}$

$$\Rightarrow \psi(x,y) = A e^{ik_x x} e^{ik_y y}$$

$$= A e^{i\vec{k}\cdot\vec{r}}$$

$\vec{k} = k_x \hat{x} + k_y \hat{y}$

$\vec{r} = x\hat{x} + y\hat{y}$

Time dependent displacement:

$$\psi(x,y,t) = \mathrm{Re}\left(A e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t}\right)$$

$$\omega^2 = \frac{4T_H}{m a_H} \sin^2 \frac{k_x a_H}{2} + \frac{4T_V}{m a_V} \sin^2 \frac{k_y a_V}{2}$$

(k_x, k_y are arbitrary for the moment.)

In general:

2-d problem can be infinitely hard!!

(In this special case, it is solvable.)

Before introducing the boundary (infinitely long system)

① Normal mode in 1D:

Always two normal modes: $e^{\pm ikx}$

② But in 2D:

In general, a fixed ω

$\Rightarrow$ infinite number of solution

(if we lower k_x, we can alway increase k_y a bit to compensate!)

Adding walls back in:

Boundary conditions gives

$$K_x = \frac{n_x \pi}{L_H} \qquad L_H = 5a_H$$

$$K_y = \frac{n_y \pi}{L_V} \qquad L_V = 5a_V$$

$$\psi_{n_x,n_y}(x,t) = A_{n_x,n_y} \sin\left(\frac{n_x \pi x}{L_H}\right) \sin\left(\frac{n_y \pi y}{L_V}\right)$$

General solution: linear combination of ψ_{n_x,n_y}

Yen-Jie's cheat sheet for demo: soap 2D waves:

57 Hz, 94 Hz ~~129~~ 139 Hz