PROFESSOR: So, I'm back. Welcome back, also, to 8.03. So today, what we are going to do is something really interesting. It's to understand how we use symmetry to help us with prediction of physical situations. So first, I will go through two concrete examples of symmetry, and see what we can learn from there. And also, today, we are going to go to infinite number of coupled oscillators. OK? I think we are done with finite numbers. OK?

All right. So what we have learned last time when Bolek was giving lectures, I hope we have learned that driving force can excite a specific normal mode. Right? So if you drive the system at the frequency, the system like, then the system will respond, and will oscillate the driving frequency with large amplitude. OK?

And also, we have learned that the full solution of a coupled oscillator is actually pretty similar to the situation we got from single oscillators. So that where you have a particular solution, and a homogeneous solution. And the full solution will be a superposition of the two component, and all the unknown coefficients in the homogeneous part of the solution. OK?

And today, I hope I can help you and convince you that symmetry actually can help us to solve the number of modes without knowing the detail of M minus one K metrics. So that actually sounds really cool, and I would like to talk about that in this lecture today.

So this is actually what we have been doing so far. So we tried everything in terms of metrics. So we start from the equation of motion, and X double dot, you go to minus KX. And then we write everything in a complex notation-- exponential i omega t plus phi times A-- A is actually the vector, right? So it's actually A1, A2, A3. It's actually the amplitude of the oscillation of the first, second, and third and etc. etc. it's a component of the system. Right?

Then, we actually found that, in the end of the day, we are actually solving this problem like eigenvalue problem. So basically, we have M minus 1 K metrics describe how each component in the system interacts with each other. OK? Then, what is actually the angle of frequency of the the normal modes? Essentially, coming from this eigenvalue problem , M minus 1 K, A equal to omega square A. Then you just go ahead and solve the eigenvalue problem. Then you will be able to figure out why are there no more mode frequencies, and therefore, what are the relative-- the ratio of the amplitude in the normal mode, which is actually the A vector. OK? The eigenvector.

OK, so that's actually what we have been doing. OK? And today, what I'm going to do is to introduce you a very important concept in physics. Not only in physics, but also in mathematics, and also art, right? So you see symmetry in art, for example. We can see here-- there are several graphs here-- and you can see that their apparent symmetry, or rotational symmetry, they are refraction symmetry. And you can see that when we build the particle detector for example lower right plot is a CNS detector in the Large Hadron Collider.

We also try to build this detector symmetric, right? Because otherwise, if we get a very complicated shape of detector, then the analysis of the data will be really complicated. So therefore, everybody like symmetry, and everybody don't like, really, chaos. Right? OK? So, that's really nice.

The question is, how do we speak the language that the nature speak? How do we actually describe symmetry? That's actually the question I'm asking, and I'm going to show you that, OK, we can actually use mathematics to describe symmetry. So before we go to infinite number of oscillators, let me give you a concrete example of symmetry, and then see if we can understand how to use the math to describe symmetry. OK?

So there is a two-component system. Two pendulums, which we worked together in the last few lectures, that they are coupled to each other, and there's a parent symmetry of this system. Can somebody tell me what is the symmetry, you can see from this system? Somebody? Anybody?

AUDIENCE: Reflection.

PROFESSOR: The reflection symmetry. So if you reflect this system, as I show you in the slide, you can see that if you reflect this picture, it looks identical. Right? So that is actually really, really good news. That means if I do this reflection, XY, and go to minus X2-- you have a minus sign, because you can see that after reflection-- the amplitude changes sign. Right? X2 go to minus X1, the system looks identical, and the physics should not change. OK? So that's actually what we can learn from there.

So that means if I have-- I do this reflection, then I can actually define X tilde-- T-- this is equal to minus X2 minus X1. OK? To become paired with X. OK? And this is also going to be the solution of the equation motion if the original X is already a solution. OK? So that's the power of reflection symmetry. OK? If X is a solution, then I do this reflection, and I can figure out that X tilde is also a solution. OK?

So how do I actually describe the symmetry in the form of mathematics? What we actually do is to define S matrix, symmetry matrix. And in this case, when we talk about reflection symmetry, it's actually defined as zero minus 1 minus 1, 0. This is actually a two by two matrix. And if I do this operation, S operate on this X matrix, then that is actually is going to give you the X tilde. OK? So that's the nature of the role of the symmetry matrix. OK? Any questions?

OK. So now we have defined a symmetry matrix. And then you can ask, why do we actually care, and why do we actually introduce symmetry matrix? Right? Because I can always write down the X tilde in that way. That is because I think by the end of this lecture, you will find that if S matrix describes the symmetry of the system, OK, that would mean S matrix will commute with M minus 1 K matrix-- which, we don't know commute yet, but I will introduce you-- that means M minus 1 K matrix and S can actually swap freely. OK? If that happens, then S matrix will share the same sets of eigenvectors as the M minus 1 K matrix.

What does that mean? That means-- OK. Before we are doing this solution, right, we are solving M minus 1 K matrix eigenvalue problem, right? And then, we get the eigenvector, which is the amplitude ratio of normal modes. And that means you have an alternative way to get the normal mode. You can solve the eigenvalue problem of S matrix, then you can get the same set of amplitude ratios as M minus 1 K matrix eigenvalue problem. OK? And then usually, the eigenvalue problem of S matrix is far much easier than M minus 1 K matrix. OK? So that's actually why we're doing this. OK?

So now, I would like to convince you that S matrix and M minus 1 K matrix will share eigenvectors. OK? So. Let's go ahead and prove this, or demonstrate this idea. OK? So the original equation of motion looks like this. X double dot equal to M minus 1 K X. Right? So now, this is actually the original equation of motion. And if this system satisfy the reflection symmetry, that means X tilde is also a solution, right? Therefore, what does that mean? That means X tilde double dot will be also equal to M minus 1 K X tilde. Because it's also a solution to the equation of motion, right? That's pretty natural. OK?

Now. I can actually use this expression, X tilde is equal to S times X. Right? All of those things are matrix, OK? Just to be careful. That means I can write this like this-- S X double dot equal to M minus 1 K S X. OK? There's no matrix, and I also replace-- I'm just replacing X tilde by S X. OK?

And also, I call this, actually, 1; I call this actually 2. OK? I can multiply X from the left-hand side of 1. OK? And see what will happen. So if I do that, then what I am going to get is S X double dot-- OK? That will be equal to S M minus 1 K X. OK? If you compare this equation, and the equation number three, these two equations, you will see that let-hand side is the same. Right? Right-hand side-- huh! Something interesting is happening. M minus 1 K S must be equal to S M minus 1 K. What does that mean? This means that they are the same. M minus 1 K S is actually equal to S M minus 1 K.

So if I say, this distance satisfy a symmetry described by S matrix, that means X tilde, which is actually the transformed amplitude, will be also a solution to the equation of motion. And therefore, an inevitable consequence is that M minus 1 K S will be equal to S M minus 1 K.

Usually, when you started physics, we write this in terms of commutator. OK? So we call this, these two things actually commute. OK? So commutator is actually defined as A bracket of A and B. This is actually equal-- defined as A B minus B A. OK? If A and B commute-- OK? It's this new word, probably, for most of you-- if they commute, that means A B in the bracket is equal to zero. OK? So this expression, I can actually write it down like this. Commutator of S M minus 1 K, that is equal to zero. And you will see this really a lot when you study quantum physics. OK? So I hope this actually gives you some flavor about commutator. OK?

So now, that's actually pretty nice. This means that they commute, OK? If I take X of t this is equal to A 1 cosine omega 1 t. OK? So, this means that A is actually-- sorry, X is actually a solution, which is a normal mode, a solution. Right? And A is actually amplitude the vector, the amplitude vector of the first normal mode, and omega 1 is actually the first normal mode frequency. OK?

If this is the case, then I will have X tilde of t will be also oppositional to A 1 cosine omega 1 t. Because if I actually exchange X1 and X2, the oscillation frequency is not going to change. Right? Therefore, since this system is in the same normal mode with angular frequency omega 1, therefore the amplitude ratio of the first and second oscillator will stay constant. Right? Because you are in one of the normal modes. Right? Therefore, I can conclude that X tilde is going to be proportional to this expression. Because they are in the same normal mode, oscillating at the same frequency. OK? Is that too fast? Everybody is following? OK.

So that's nice. So this means that S X of t will be equal to S A 1 cosine omega 1t, OK? So this is actually coming from here, right? I am replacing X tilde by S X based on this definition. OK? Then again, I replace, I write, X explicitly which is actually A cosine omega 1 t. OK? Then you get this expression. And from this expression above, you see that you conclude that this is proportional to A 1 cosine omega 1 t. That's very nice. That means S A cosine omega 1 t is proportional to A 1 cosine omega t. And you can actually cancel this. And you see that S A 1 is proportional to A 1. Or I can write it as S A 1 is equal to beta A 1.

What does that mean? This means that A 1 originally-- where's A 1 coming from? A 1 is the amplitude of all the components in the first normal mode. Right? That's coming from the eigenvalue problem, which it actually does in this light. Eigenvalue problem M minus 1 K A equal to omega square A will give you the solution of normal mode and their eigenvectors, which is amplitude ratios of all the components in the system. Right? So that means A 1 is not only M minus 1 K matrix eigenvectors, it's also eigenvector of S matrix. OK?

So that is actually very good news. And I can also do the same thing for A 2, to prove that it also works for A 2-- the derivation is identical, so I am not going to do that again. So that means, actually, starting from here, OK-- if X and X tilde are both solutions to the equation of motion. I will conclude that S matrix and M minus 1 K matrix, they commute. OK? How to tell if a system satisfy a specific symmetry defined by my symmetry matrix? Is by this way, you can check if M minus 1 K and S commute. If they commute, that means the system actually satisfy this symmetry. And also, the consequence is that from there, you will conclude that if you have also a set of eigenvectors from M minus 1 K matrix eigenvalue problem, then that is going to be also the eigenvector of S. OK? Any questions?

OK. So M minus 1 K eigenvectors. Also S eigenvector. OK? That's actually what we have learned from this small exercise. Now, you can say, wait, wait, wait, wait. This is actually not what we need, right? I would like-- we would like to argue that S matrix-- I can solve S matrix eigenvalue problem, and I can learn about the solution of M minus 1 K matrix, right? This logic is actually in the opposite direction, right? You said, OK, you solved things already, then, actually, it's also S matrix eigenvalue problem. So now what I am going to do is to reverse the logic, and see if it works. OK? Again, to see what will happen. OK?

So now, I would like to prove that if I solve S matrix eigenvalue problem, I have also solved the eigenvectors for M minus 1 K matrix. Run the logic in the opposite direction. OK? So, if I were given two things-- one, S A is equal to beta A. Number two, S matrix and M minus 1 K matrix commute. OK? If those are the given conditions, then I can actually conclude that S M minus 1 K-- OK? I can actually contract this expression-- I write that S M minus 1 K A, OK? Because they commute, right? They can actually swap M minus 1 K and S safely without actually introducing any more terms. This will be equal to M minus 1 K S A. OK? And S A, from the first expression, S A is equal to beta A. Right? Beta is a number, OK? Therefore this expression will become beta M minus 1 K A.

So, beta can penetrate through matrix, because beta is just a number, is eigenvalue. It's eigenvalue of S matrix. OK? So what does this mean? OK. So what does it mean? So this means that if you look at this part and that part-- you look at the beginning and the end of the expression-- you immediately conclude that M minus 1 K A, this expression is also an eigenvector of S matrix. Right? So you have S matrix acting M minus 1 K A. And that will give you something proportional to M minus 1 K A. You see? It's magic, right? It's actually not magic, but it's actually just, you know, really logical extension. Right? OK?

Very cool! So that means this is also an eigenvector of S. Right? And also, another thing which is interesting is that they share the same eigenvalue, beta. Right? They have the same eigenvalue. OK? So, if eigenvalues of S-- so you can get several eigenvalues, right? In this case, two by two matrix, you will get-- how many? Two, right? Two eigenvalues. If those two eigenvalues are different, then I can conclude that M minus 1 K A must be proportional to A. Right? Because this is actually the same eigenvalue problem, and the same eigenvalue, beta. Since all the eigenvalues from the solution of eigenvalue problem of S A equal to beta A, those eigenvalues are all different, therefore I can argue that M minus 1 K A is proportional to A. OK?

Therefore, M minus 1 K A is equal to omega square A. Omega square is actually some constant. OK? This is actually amazing, because that means given the two conditions-- the first one, I can figure out the eigenvalue and the eigenvectors of S matrix; second, if S matrix and M minus 1 K matrix interaction matrix, they commute-- then I can actually already figure out what are the eigenvectors of M minus 1 K matrix. OK?

And another thing which we've learned from here is that, wow, that's good! Because the eigenvectors are already solved. Therefore, I just have to calculate this. It's just a normal operation. It's not the eigenvalue problem anymore. I just multiply M minus 1 K times A, then I can actually get the value omega square. You see? That's actually much easier than solving the eigenvalue problem of M minus 1 K matrix. OK? That's actually very good news.

Finally, I think the most important consequence is that once we solve this system, which satisfy the symmetry described by this S matrix, we have solved all the possible systems which satisfy the same symmetry. For example, in this case, I solve a coupled pendulum problem, OK? They look symmetric. Right? And I can, of course, I can draw another one, which is like this. It's more circular. And there are two walls, which is actually-- there are three springs connected to the wall. This problem is already also solved, right? Because it also satisfy the same symmetry.

And of course-- like, you know, like this, go crazy, and even more. This is also solved! Right? Because this is also symmetric. Right? I can add more. Right? Like this. This is also symmetric. Right? And this-- let's think. The eigenvector of this M minus 1 K matrix eigenvalue problem will be identical to what we have already solved here. OK?

So, that is actually really amazing. If you speak the right language, and cut into the problem in the right angle, you actually find that actually, you can solve multiple problems at one time. OK? Any questions?

OK. So now this is actually very nice, and this is actually a very important preparation to the next step, actually. So now, we have understood coupled oscillator, and we have learned a little bit about symmetry. Therefore, I would like to go to infinite number of coupled oscillator. OK? So that is actually the next step, which we are going to move on in 8.03

So this is actually one example infinite system. OK? I cannot write the whole universe. Why? Because it's infinite, so I couldn't include everything in the slide. But this is actually an example system. Done OK? Looks hopeless, right? In general, we don't know how to solve infinite system, because if you have infinite number of things that are connected to each other in random ways, then the problem becomes really, really complicated. OK? In general, I don't know how to solve this problem. And if you are a EE major, the first thing, maybe, you like to do is, ah, now I have this picture, and I can put everything in my computer, and see how things evolve as a function of time! Right? Of course we can rely on the computers, and see what we can learn from it. And if you made your major of mathematics, you will say, no, this is not the problem I am going to work on. OK? I don't care.

But as a physicist, what we are going to do is that, huh-- we look at this infinite system, OK? It's kind of interesting, right? It's a lot of things, a lot of small balls connected to big balls, right? Super big ones, and plotting things in log scale. So those balls are really, really large compared to all the other balls connected to this system. Therefore, as a physicist, I'm going to ignore all the other balls. Oh, if I do that, then it becomes-- there is some kind of symmetry you can actually see from here, right? What is actually the symmetry? you see? There are three balls that connected to each other. They are equally spaced. We have a translation symmetry. You see?

So you can see that, actually, that's how we think about a problem. Of course, different field have different kind of thinking, and different kind of problem they would like to focus on. But as a physicist, I would like to know how the system will work, and that is actually what I'm going to do. OK?

So that's very nice. We are going to discuss infinite system. So what is actually the infinite system I am going to talk about? It's actually there is infinite system with space translation symmetry. So, to save some time, I have already written down the matrix involving this system here. What I am interested is mass sprint system, OK? Infinite number of mass and spring. And they actually satisfy space translation symmetry. OK? They are connected to each other by springs, with natural length A and spring constant K. OK? And there are infinite number of them, actually, lined up from the left-hand side of the edge of the universe to the right-hand side edge of the universe. OK? I've prepared this system. OK? It took me a long time. OK? All right?

But it's very difficult to describe this kind of system, right? So the first thing we have learned from 8.03 is that in order to describe this system, I need to define a coordinate system, right? And also have everything properly labeled. So I introduce a label-- j minus 1 j, j plus one, j plus two-- just to name each little mass I'm talking about. OK? No other purpose. Then, once I have the label, I can actually write everything, express the displacement of little mass, as X j minus 1, X j X j plus one, X j plus two. That's just the displacement from the equilibrium position of the mass. OK?

And this system will have equation of motion looks like this. So if now I focus on the little mass, Z. OK? Then I can actually write down the equation of motion. There are two springs connected to these mass. Right? Therefore, you are going to have two spring force. Right? Since this is actually idealize the springs with spring constant capital K, therefore, I can write down immediately the equation of motion is actually equal to M X double dot j is equal to minus K X j minus X j minus 1 minus-- this is actually the right-hand side spring force-- minus K X j minus X j plus 1. We have done this exercise before, right, with a simpler problem. OK?

As usual, I can collect all the parents associated with X j minus 1, X j, and X j plus 1, together. Then I get this expression, which actually looks nice. OK? And I assume that this system is actually undergoing some kind of oscillation. OK? Therefore, I assume that this solution, X j will be equal to A j is the amplitude of j's mass. OK? Cosine omega t plus phi, omega is actually the oscillation frequency, and phi is actually the phase, and I don't know why this is actually omega and A j yet. OK? We would like to figure that out. And as usual, you can actually write down the M matrix, OK? M matrix is actually really simple, in the diagonal terms-- diagonal terms are all m, and the off diagonal terms are all zero. Right? And you don't really need to copy them, because they're all derived in the lecture notes.

M minus 1 K matrix-- ha! I have already arranged my terms here; therefore it looks like this. It have a strange structure, you have three terms, kind of in the diagonal terms, and this actually is shifting as a function of number of rows, and all the other parts of the matrix actually zero. OK? It's an infinite times infinite dimension matrix.

Finally, I would like to also write my A matrix is the vector of amplitude, right? So you have many, many numbers-- A j, A j plus 1, A j plus 2. OK? And et cetera, et cetera. OK? Now, very easy, right? The question is actually can be solved, right? You just have to solve the M minus 1 K matrix, right? That's easy, right? It's an infinite number times infinite number matrix, right? Super easy! No, actually not. Right? [LAUGHTER] So we are in trouble. I don't know how to solve this problem. OK? What can we do? Anybody have any suggestion to me?

AUDIENCE: Ask the math department?

PROFESSOR: Ah, yeah! Math department is coming in to help. Yes. But actually, before asking them, we learn some concept, which we just learned, right? This-- what kind of property of this system?

AUDIENCE: Symmetry.

PROFESSOR: Symmetry! Right? We have symmetry. OK? So this M minus 1 K matrix looks horrible. But if I write down the symmetry matrix, actually, it looks slightly better. OK? So what is actually the symmetry matrix? So one observation we can make from this system is that if I shift this system, A, to the left, OK? I shift these two mass to the left-hand side, I shift all the mass. I have to hire many, many students to move all the mass from left-hand side of the universe to right-hand side of the universe. OK? And after they have done that, the system looks the same. Right? That's very good, OK? After all the hard work, right?

So what is actually going to be the symmetry matrix? OK. Now, I would like to achieve something which is A prime equal to S A. And then this S actually shift the mass by a distance of A. Right? So what would be the functional formula for this S matrix? It would look like this. It's going to be 0, 1, 0, 0, 0-- 0, 0, 1, 0, 0, 0-- looks like this. OK? So the next two diagonal term is all one. All the rest of the component is zero. OK? And this looks a lot more friendly compared to M minus 1 K matrix, right? Still, this is horrible thing to do, because this is infinite number times infinite number dimension matrix. OK?

So. We would like to find the eigenvectors of S matrix. OK? So this means that if I manage to solve the eigenvalue problem, assuming that-- OK, I haven't solved it, but assuming that I can solve it, then what I'm going to do is going to get this S A will be equal to beta A, where A is actually a eigenvector of S matrix OK? And S A, we just learned from here, is actually equal to A prime. So beta is the eigenvalue, and A is actually the eigenvector.

So that means, originally, I have A, which is something something A j, A j plus 1, A j plus 2, blah blah blah. OK? And A prime, after I actually multiply A by S matrix, I get A prime, which looks like this-- A j plus 1, A j plus 2, A j plus 3. OK? So what I am going to do is-- what, actually, this S matrix does is to shift the A component one row, right? OK? So then, we basically get this expression. And of course, A 1 is equal to beta, which is a constant, times A. Right?

So if you compare, for example, here, you can get that-- A j prime will be equal to beta A j, which is actually equal to A j plus 1. Right? A j prime is actually equal to A j plus 1, right? It's just shifting one unique label. Right? OK.

So this is actually the expression I'm looking for. OK? We don't know yet why this is actually beta. Beta is a number. Assuming that I can solve the eigenvalue problem. OK? But I do know, if I have A 0, if A 0 is equal to 0, from this expression, that means A 1-- sorry, A 0 is equal to 1. If A 0 is 0, then everything's 0, right? And it's not fun, right? OK. A 0 is equal to 1, then something will happen. A 1 will be equal to beta, right? From this expression, right? Because beta A j is equal to A j plus 1, A 2 will be equal to beta square, et cetera, et cetera. And then I can say that A j, if I assume A 0, if A 0 is equal to 1, then A j will be equal to beta to the j. OK?

Am I going too fast, here? Everybody is following? No questions? No? Good.

Actually, we found that we have already solved the eigenvalue problem. Right? Because I have already the expression for the A j, which is actually in the form of beta to the j, right? So beta is some kind of number, and the infinite number of beta actually can satisfy this eigenvalue problem. No matter what kind of beta I choose-- it can be 1, it can be 2, 3.14, it can be pi-- and what am I going to get is the corresponding A j, corresponding A vector, which you have satisfied this expression. OK?

So that means some magic happen. We have already solved the eigenvalue problem without really deriving, you know, a lot of deviation. Right? Secondly, another thing which we learned is that there are infinite number of eigenvalue which satisfy this eigenvalue problem. The question is, does that make sense, or not? Infinite number of eigenvalues can actually satisfy this infinity long system. It's kind of making sense, right? Because we have worked on one oscillator, you had one normal mode; two oscillator, you have two normal mode; three oscillator, you have three normal mode-- infinite number of oscillator, you should have infinite number of normal modes. Right?

OK, so that is actually a very, very good news, because we have already solved the problem, and we also know the function of four of eigenvectors. OK? So let's take a look at those example system, which are actually close to infinity long.

So here, you have a Bell Lab machine, which actually can have, actually, multiple coupled oscillators. Each one of them can oscillate up and down, and you can see that, huh, if I actually tried to move them up and down, that a complicated kind of motion can occur from this system. Actually, if I do this, you see that, ah, they are something similar to wave is happening. And if I do this continuously-- oh, some kind of, like, a standing wave is produced, right? And this system is actually really, really hard to describe, right? If you look at how many things this system can actually do. OK?

Another example is actually-- OK, so you can say, come on, this is actually not infinitely long system, right? You have some final number, right? So how about I use this system as a demonstration. This is actually a much nicer, or much better, approximation, OK, to infinitely long system. You can see that, OK, each mass, each-- OK, I can say, for example, each small component of the spring, essentially, can become seeded as a small m in my graph, right? And actually, I can, instead of oscillating them back and forth, I oscillate them upside down.

OK?

And you can see that, huh, they are interesting kind of motion. I can have-- I can have this, which is like a standing wave; I can do this; I can stop this system, and I produce-- woo! I can produce a wave. And then it goes back and forth. And I can, whoa, do this crazy, and then you see that, how exciting-- a much higher frequency normal mode, right? And that's really complicated. And the question is, how can we actually understand this kind of system? The thing is that this system is so much, so complicated, and have infinite amount of possibilities. Right?

So how are we going to understand this? Very good news is that we have solved the normal modes of this kind of system, right? So the normal mode looks like this-- A j equal to beta j. OK? And the following lecture, the rest of the lecture, is to understand what does that mean, and also make predictions. OK?

So now we have, actually, the eigenvectors, OK? That's really nice. So from our previous discussion, if this system actually satisfy the symmetry, have the symmetry that is acquired by the S matrix, which I have here, that means M minus 1 K matrix will share the same set of eigenvectors as S matrix. So what is actually part of the work is to evaluate this. M minus 1 K multiplied by A, and that will give you omega squared A. OK? So I just need to multiply M minus 1 K matrix by A. What is A? A is actually here. Now what is actually M minus 1 K matrix? M minus 1 K matrix is here, have a kind of complicated structure. OK?

On the other hand, if I only focused on the jth object, the object which is named j, have a label j, then actually I can write down, OK, the right-hand side is actually just omega square A j, right? Because this is actually-- if I only focus on the j component, OK, left-hand side is actually just M minus 1 K A multiplied by A, right? OK, so basically, there are only these three terms coming into play, right? If this is A j minus 1, so anything minus 1, we are multiplying by minus K over n. A j we multiply by 2 K over n, and A j plus 1, we are multiplying by minus K over n, right? The rest of the terms are all 0. OK? It's actually not as complicated as we thought. OK?

So, if I write it down, explicitly, the left-hand side part, then what I'm going to get is minus K over n, capital K over n, A j minus 1, plus 2 capital K over n A j minus capital K over n, A j plus one. OK?

So this is actually the j term. Now I can define omega 0 square equal-- is defined as capital K over n. If I do that, then basically, I can see that omega square A j will be equal to omega 0 square. OK? I am taking all the K over n out of the game and write it down as omega 0 square. OK? Minus A j minus 1 plus 2 A j minus A j plus 1. OK?

And also we know, from the previous discussion, S matrix and the n minus 1 K matrix should share the same sets of eigenvectors. Therefore, I can actually try to plug in one of the eigenvectors from S matrix. Right? A j equal to beta j. OK? I can plug that in, then basically, I get omega 0 square minus b-- minus beta, j minus 1 plus 2 beta to the j minus beta to the j plus 1. And the left-hand side will be reading like omega square beta to the j. OK? Questions? OK.

So now, I can cancel-- I can actually divide everything by beta to the j, right? I can get rid of beta to the j, then basically, I get omega square equal to omega 0 square minus 1 over beta plus 2 minus beta. OK? And as we discussed before, beta can have any value. OK? And also, you can see from here that, huh-- once I know the eigenvalue of S matrix and eigenvector of S matrix, I also know what is actually the corresponding angle of frequency of the normal mode. Right? By using M minus 1 K times A, you can figure out what is actually the corresponding omega, the normal mode frequency. OK? So that is actually pretty nice.

But on the other hand, if you step back and just think about what we have been doing, OK? So very good. You have a beta, which is a random value. You can evaluate this thing, then you can get the corresponding omega. But then something doesn't feel right. Right? For example, if you have beta equal to 2, what is going to happen? If you have beta equal to 2, what does that mean? That means A j will be equal to 2 to the j. OK? That's very dangerous. Hey?

That means-- OK, so I am-- I deploy the whole system, OK, from the left-hand side of the universe to the right-hand side of the universe. OK? So that means, if I go to the your right-hand side of the universe, the amplitude explode. Right? It's actually 2 to the infinite number, right? OK? It's not a physical-- doesn't sound like a physical system to me. Right? If, actually, beta is greater than 1, then the right-hand side A of the universe, the amplitude there, will explode. OK? Doesn't sound right, right? So I don't like that. OK? Maybe you like it, but I don't like it. For the moment.

On the other hand, if the beta-- OK, again, it's not 1, but smaller than 1-- what is going to happen? If the beta is smaller than 1, what is going to happen is that, huh, OK, the right-hand side of the universe is fine, is finite, because the amplitude has become smaller and smaller. But the left-hand side part of the universe, the amplitude still explode. Right? So what does that mean? This means that if beta-- if the absolute value of beta is not equal to 1, the amplitude, at some point, goes to infinity. OK? So that's actually not very nice. That's because A j is actually proportional to beta to the j. OK?

So in the discussion we have here, we consider beta equal to 1 case. OK? Otherwise, it's actually, things will explode. OK? So if the absolute value f beta is equal to 1, in general, beta can be exponential i, small k A. Right? Then, actually, you can get absolute valuable of 1. OK? If beta is equal to 1, that means the amplitude of all the oscillators are the same. OK?

All right, so now, if we accept this, we only limit ourself to the discussion of beta, absolute beta, value of beta equal to 1, then beta can be written as exponential i k A. Then, if I plug this back into this, basically, what you are going to get is omega square is equal to omega 0 square 2 minus exponential i k A plus exponential minus i k A. Right? Because you have minus 1 over beta, and beta, therefore you have exponential i k A, and the exponential minus i k A. OK? It's a lot of math in this lecture, but we are getting over to it. OK?

All right. So that is actually-- we actually can identify this, and this actually can be rewritten as 2 omega 0 square 1 minus cosine k A. OK? We have arrived a surprisingly simple expression. So let's take a look at this expression carefully. So that means, for each given k, a small k, then I will have a corresponding angular frequency, omega square. OK? So still, there are infinite number of possible normal modes. OK? From this. So if I take a look at the amplitude, if I select a k value-- small k value-- if k is given, I can actually calculate the corresponding A j. So the A j I can actually define as a superposition of exponential i j k a, and minus exponential i j k a. And that will give you a sinusoidal shape.

So if I give you the k, basically, you'll see that if I give you a k, then you get the corresponding beta. Right? And you are going to get omega, the corresponding omega square. But one interesting thing of this expression is that if you keep beta, or keep one over beta, you are going to get the same omega. Therefore, I can now use superposition principle. Basically, I can actually add these two solutions together, since they are going to be oscillating at the same frequency. Then what I'm going to get is, huh, interesting thing happen. The A j, the amplitude, as a function of j, it's like a sinusoidal function. OK?

So that is actually what is really predicted to an infinity long system. For example, if I do this, you can see that, aha, indeed, I can see sinusoidal shape. OK? And you can see that the sinusoidal shape is actually oscillating up and down, like a standing wave. And that is actually exactly this expression. So that tells you something really interesting. That means the sinusoidal shape is associated with what? Associated with translation symmetry. Right? All I have been doing is to require this translation symmetry, and you already get the amplitude A j. And if you choose the physical beta value, then you already immediately arrive at a solution which is actually like sinusoidal shape. Doesn't that sounds really amazing to you?

OK. So I think it's time to take a five-minute break, because I can see that you are overwhelmed by the math already, and of course, let's come back in five minutes, then we can discuss some more about what we have learned from this mathematics. And if you have any questions, please let me know.

OK, so welcome back, everybody. Of course, you are welcome to come back here, and play with the demonstration. OK? So very good. So during the break, there are several questions asked, which I think, those are very good questions, and that's actually the purpose of this break. So it's a long day already, right? A lot of mathematics, and I hope everybody survived. OK? No dead body yet?

You can see that here, I'm doing something really crazy, here. So, OK. Consider-- I think most of you got this point, beta not equal to 1 is not nice. Something explode at the edge of the universe. So I don't like that. Therefore, I consider only the case which you have absolute value beta is equal to 1. And then we say, OK, it can be plus 1 and minus 1, but that's actually not the whole story, right? You can have, in general, beta equal to exponential i, some number. Right? Some real number. OK? And I write, here, a very fancy expression. Beta equal to exponential i k a.

Why i k a? It's a very good question, right? What is a? I think most of you actually already forgot. What is a? a is actually the natural length of the spring. OK? So I was going too fast, because I would like to get to a break to hear your questions. So what is a? a is the natural length. OK? And the k-- what is k? Later, you will figure that out. You'll find that, actually, k is a wave number. OK? So that is actually much more of meaningful now, right? After the explanation. So you can see that beta is equal to exponential i, some number, and I call it k a, a fancy name of this number, and it has some physical meaning. OK?

Another thing which is interesting is that if I plug in beta equal to a, or beta equal to 1 over a, into the same expression-- if I plug in either beta a or beta equal to 1 over a to this expression, I'm going to get exactly the same omega. So that means, OK, both of them will be-- both value will be oscillating at the same frequency. OK? So if you choose beta equal to a, choose beta equal to 1 minus a, they are oscillating at the same frequency.

What does that mean? That means linear combination of eigenvector coming from beta equal to a and eigenvector coming from beta equal to one over a, linear combination of those eigenvectors are also eigenvectors of the M minus 1 K matrix. OK? And that's actually where-- OK, those are different eigenvectors for S, but the linear combination of these vectors are all the-- eigenvector of M minus 1 K matrix and always the same eigenvalue omega square. OK? So that's another thing which is important.

And finally, I said that there are infinite number of choice of k. That's valid, right? Because you can choose a little number, then you get a corresponding beta, then you get a corresponding omega. So you have infinite number of normal modes. Secondly, if I give you a k, OK-- if I give you a k, or I can give you another value which is minus k, then that means you will get beta and 1 over beta. Right? Minus k will give you 1 over beta. Right? And as I mentioned before, beta equal to a and beta equal to 1 over a will give you the same omega. Therefore, a linear combination of the vectors are also eigen of M minus 1 K matrix.

Though, that's actually what I am doing here, right? So in order to show you a real amplitude, I'm doing a linear combination of exponential i j k a, and exponential minus i j k a. It's just a choice. OK? Of course, you can say, OK, I choose plus, and divide it by 2, then you get the cosine. Right? But if I choose this expression, then what I am going to get is that, huh-- since both of them are-- both vectors are corresponding to the same eigenvalue omega square, therefore, linear combination of them also oscillate at the angle of frequency omega. Therefore, if I calculate this and make it real, then I find that the amplitude is a function of j. Is actually a sinusoidal function, which is sine j k a. OK?

So what does that mean? This means that if I plug the a-- if I plug A j as a function of j, this is actually what I'm going to get. It's a sinusoidal shape. OK? And we know that x j is actually equal to A j cosine omega t plus phi. Right? Omega, I can actually evaluate that, right? From here, right? Just a reminder. And what we are going to get is, when this system is thinking of normal mode, OK-- actually, this system is still a discrete system, so i-- actually, would like to point out that as a function of j, only discrete location have mass. Right? So you see that those are individual mass. They are oscillating up and down. OK?

And you can see that, OK, since they are oscillating up and down, therefore, the oscillation, essentially, going up and down. Therefore, what is the actually the normal mode of this infinity long system? The normal mode are actually standing waves. But they actually only appear in the discrete value of j. And it has a functional form of something like a sinusoidal shape, or cosine. OK? So that's actually what we learn, and actually, you can see that from here. So if I oscillate this at some selected amplitude-- OK? Not quite get it. Yeah. So you see that, OK, it's roughly like a standing wave. It's a fixed frequency. OK?

I would like to discuss with you a really interesting selection. So if I now take a look at-- so we have went through a lot of math, right? So now is the time to enjoy what we have learned, right? So if I now take a extreme value, cosine k a, OK, equal to minus 1. OK? Then I am reaching the maximal oscillation frequency. Right? So if I choose cosine k, small k, a equal to minus 1, OK-- what is going to happen is like this. It is as a function of j, by product A j is a function of j, what you are going to get is starting like this. Those are actually the amplitude of individual mass. So you can see that if cosine k a is equal to minus 1, omega square, based on that expression-- 1 minus, minus 1, you get 2-- therefore, you get omega square equal to 4 omega 0 square. OK?

And if you plug the A j as a function of j, that is actually what you are going to get. You actually have maximal stretch to the system. Right? You can see that it's actually positive, negative, positive, negative, positive, negative. That would reach the maximum speed of the oscillation. And of course, we cannot demo-- we cannot demo maximum, infinite number of oscillator, but of course, I can demo a system with 10 oscillators. So you can see that now, I maximize the amplitude of the highest frequency normal mode. And then I let go, and you see that this is actually exactly what is going to happen when I have cosine k a equal to minus 1. Then the wavelengths-- it's very small-- and you actually reach the maximum speed. Maximum speed is actually to become paired with, for example, lower frequency modes like this one. This is actually oscillating at a much lower frequency.

And you can ask, OK, does that make sense? If I have this really, really zig-zag shape, why this system should be oscillating at the highest possible frequency? Why is that? It also makes sense, right? If you have that set up, then you are stretching this system to the maxima possible amount. Right? So, actually, now the springs looks like this. You are stretching this really hard, and therefore, the restorative force is going to be large. Therefore, you get high frequency. OK?

OK, so I hope you actually enjoy the lecture today. It's a lot of mathematics, but what we have learned is really a lot. We learned how to actually describe system, how to actually solve a system without actually touching the M minus 1 K matrix; we can actually already get the eigenvectors. And using the M minus 1 K matrix, we can actually evaluate omega as a function of the input parameter from S eigenvalue. And the next lectures, we are going to discuss more examples, and make the whole system continuous. Thank you very much, and if you have any more questions, I will be here. I'm very happy to answer your questions.