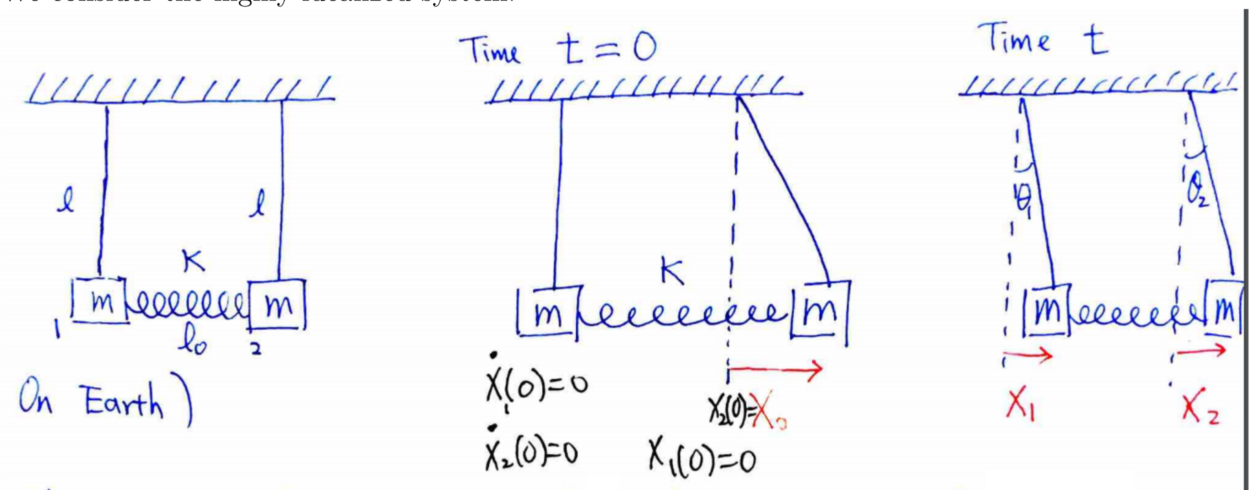
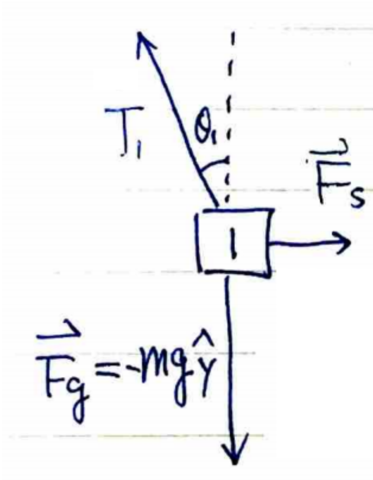
8.03 Lecture 5

We consider the highly idealized system:



Where neither block is initially moving, but the second block is displaced at a small angle at $t = 0$. There is no drag force, the springs are ideal. We want to predict the motion at arbitrary times. Define the coordinate system where $\vec{x}_1$ and $\vec{x}_2$ are measured from the equilibrium position. The $\hat{x}$ direction is to the right and the $\hat{y}$ direction is up.



$\hat{y}$ direction:
$m\ddot{y}_1 = T_1 \cos\theta_1 - mg$

$\hat{x}$ direction:
$m\ddot{x}_1 = -T_1 \sin\theta_1 + k(x_2 - x_1)$

Implementing the small angle approximation: $\Rightarrow \quad \cos\theta_1 \approx 1 \quad \sin\theta_1 \approx \theta_1$
From the $\hat{y}$ direction we get $T_1 = mg$

$$\begin{aligned} m\ddot{x}_1 &= -T_1\theta_1 + k(x_2 - x_1) \\ &= -mg\frac{x_1}{l}k(x_2 - x_1) \\ m\ddot{x}_1 &= -\left(k + \frac{mg}{l}\right)x_1 + kx_2 \\ \text{Similarly } m\ddot{x}_2 &= kx_1 - \left(k + \frac{mg}{l}\right)x_2 \end{aligned}$$

Convert everything to matrix form (recall $M\ddot{X} = -KX$)

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad K = \begin{pmatrix} k + mg/l & -k \\ -k & k + mg/l \end{pmatrix} \qquad M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$M^{-1}K = \begin{pmatrix} k/m + g/l & -k/m \\ -k/m & k/m + g/l \end{pmatrix}$$

Our equation of motion is $\ddot{X} = -M^{-1}KX$. We need to solve the eigenvalue problem. This is easiest if we switch to complex notation, define: $X =$Re$[Z]$ and $Z = e^{i(\omega t + \phi)}A$. The equation of motion becomes

$$\omega^2 A = M^{-1}KA$$

and we need to solve

$$\det(M^{-1}K - \omega^2 I)A = 0$$

$$M^{-1}K - \omega^2 I = \begin{pmatrix} g/l + k/m - \omega^2 & -k/m \\ -k/m & g/l + k/m - \omega^2 \end{pmatrix}$$

$$(g/l + k/m - \omega^2)^2 - (k/m)^2 = 0$$
$$(g/l + k/m - \omega^2) = \pm(k/m)$$

$$\Rightarrow \omega^2 = \frac{g}{l} \ , \ \frac{g}{l} + \frac{2k}{m}$$

Where we define ω_1^2 as the first and ω_2^2 as the second solution.
First examine 1: $\omega^2 = \frac{g}{l}$

$$(M^{-1}K - \omega^2 I)A = \begin{pmatrix} k/m & -k/m \\ -k/m & k/m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \quad \Rightarrow \quad A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Next examine 2: $\omega^2 = \frac{g}{l} + \frac{2k}{m}$

$$(M^{-1}K - \omega^2 I)A = \begin{pmatrix} -k/m & -k/m \\ -k/m & -k/m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \quad \Rightarrow \quad A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Go back to X: $X =$Re$[Z]$=Re$[e^{i(\omega t + \phi)}A]$

$$X^{(1)} = \cos(\omega_1 t + \phi_1)A^{(1)}$$
$$X^{(2)} = \cos(\omega_2 t + \phi_2)A^{(2)}$$

Where $\omega_1 \equiv \sqrt{g/l}$ and $\omega_2 \equiv \sqrt{g/l + 2k/m}$ as above. The full solution is then:

$$\begin{aligned} x_1 &= \alpha\cos(\omega_1 t + \phi_1) + \beta\cos(\omega_2 t + \phi_2) \\ x_2 &= \alpha\cos(\omega_1 t + \phi_1) - \beta\cos(\omega_2 t + \phi_2) \end{aligned}$$

Where the initial conditions can be used to determine $\alpha, \beta, \phi_1, \phi_2$. Implementing our initial conditions from above we find:

$$\alpha = x_0/2 \quad \beta = -x_0/2 \quad \phi_1 = \phi_2 = 0$$

Rewriting our full solution:

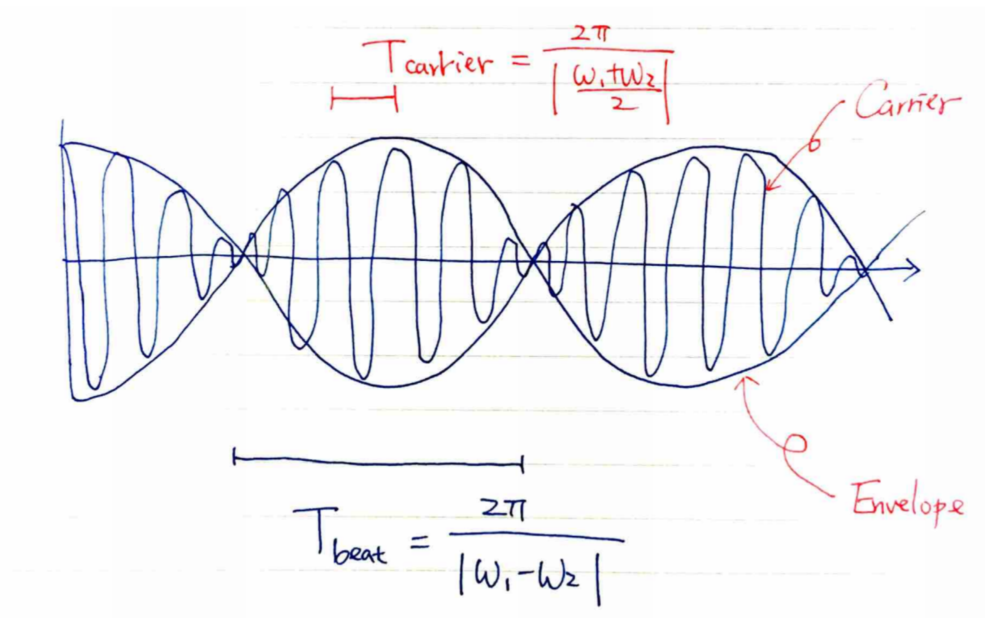
$$x_1 = \frac{x_0}{2}(\cos\omega_1 t - \cos\omega_2 t)$$
$$x_2 = \frac{x_0}{2}(\cos\omega_1 t + \cos\omega_2 t)$$

Or if we implement some trig identities:

$$x_1 = -x_0 \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right)$$
$$x_2 = x_0 \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right)$$

If $\omega_1 \approx \omega_2$ (e.g. $\omega_1 = 0.9\omega_2$)

$$\frac{\omega_1 + \omega_2}{2} = .95\omega_2 \qquad \frac{\omega_1 - \omega_2}{2} = -0.05\omega_2$$



We get two distinct waves: a carrier (high frequency) and the "beat" (low frequency) with the periods as shown.

We can define a "normal coordinate:" $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \equiv \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$

$$U_1 = 2A\cos(\omega_A t + \phi_1)$$
$$U_2 = 2B\cos(\omega_B t + \phi_2)$$

$$m(\ddot{x}_1 + \ddot{x}_2) = -\left(\frac{mg}{l}\right)(x_1 + x_2)$$
$$m(\ddot{x}_1 - \ddot{x}_2) = -\left(\frac{mg}{l} + 2k\right)(x_1 - x_2)$$

We've successfully decoupled the equations of motion!

$$\Rightarrow m\ddot{U}_1 = -\left(\frac{mg}{l}\right)U_1$$
$$m\ddot{U}_2 = -\left(\frac{mg}{l} + 2k\right)U_2$$

Where U_1 (and U_2) are oscillating harmonically at ω_1 (and ω_2)!!!!