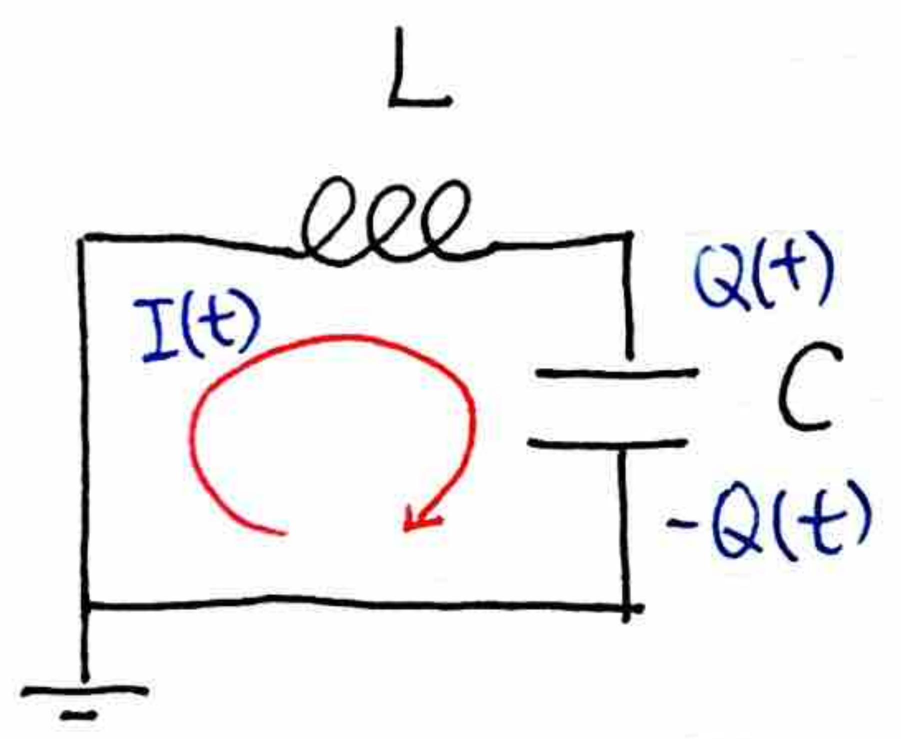
* Review of Lecture 1

✓ Another Example:

LC circuit

"Coordination System"

Define clockwise current to be positive.



$$I(t) = \frac{dQ(t)}{dt}$$

V=0 → Ground

At $t=0$ $\quad I(0) = I_{INITIAL}$, $\quad Q(0)=0$

"Initial Condition"

* Voltage Drop:

$$L\frac{dI}{dt}$$

Capacitor:

$$V = Q/C$$

$$L\frac{dI}{dt} + \frac{Q}{C} = 0 \Rightarrow \ddot{Q} + \frac{Q}{LC} = 0$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

Initial Condition

$m \longleftrightarrow L$

$k \longleftrightarrow 1/C$

$x \longleftrightarrow Q$

$$Q(t) = A\cos(\omega_0 t + \phi)$$

$$Q(t) = \frac{I_{INITIAL}}{\omega_0}\sin(\omega_0 t)$$

(From $I(0)$ and $Q(0)$ we can solve and get $\phi = \frac{-\pi}{2}$, $A = \frac{I_{INITIAL}}{\omega_0}$)

Generalized Coordinate

$$\mathcal{M} \frac{d^2 x}{dt^2} = -\mathcal{K} x$$

Generalized mass

Generalized spring const.

"Kinetic Energy" $= \frac{1}{2}\mathcal{M}\left(\frac{dx}{dt}\right)^2$

"Potential Energy" $= \frac{1}{2}\mathcal{K} x^2$

"Total Energy" $\Rightarrow$ $E = \frac{1}{2}\mathcal{M}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\mathcal{K}x^2$

If we solve the equation: $\omega_0 = \sqrt{\frac{\mathcal{K}}{\mathcal{M}}}$ $\quad \ddot{x} + \omega_0^2 x = 0$

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$\frac{dx(t)}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

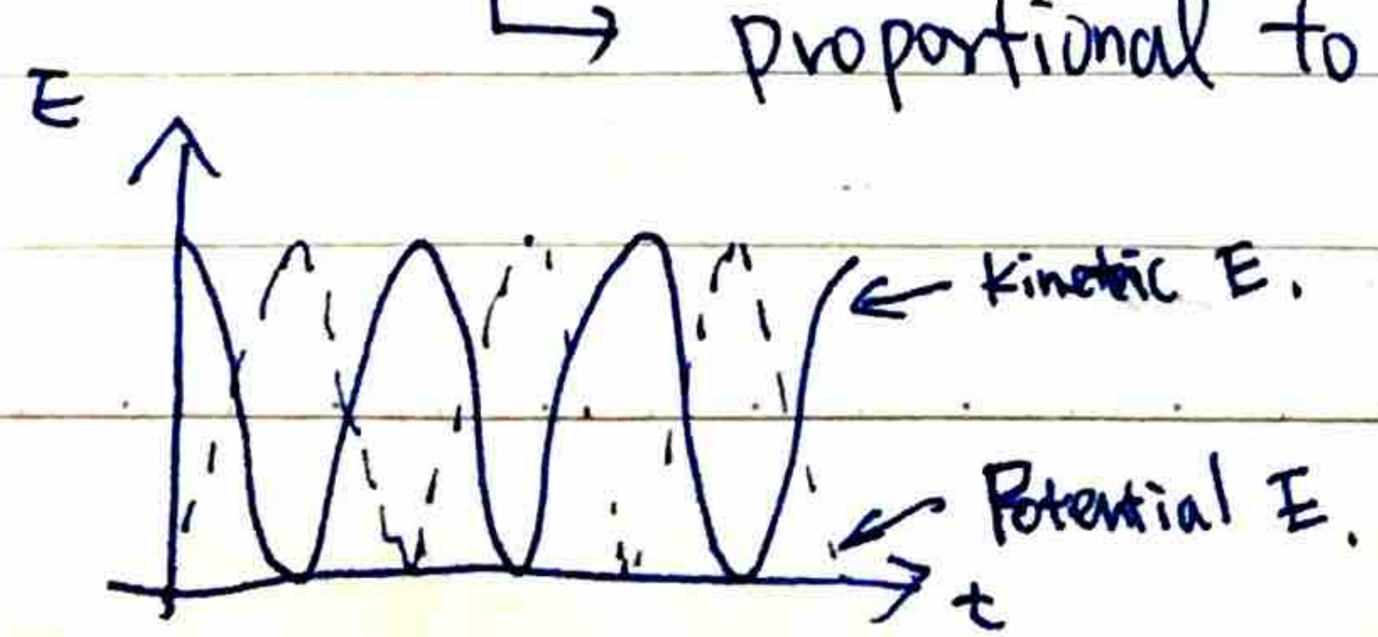
$$E = \underbrace{\frac{1}{2}\mathcal{M}A^2\omega_0^2}_{\mathcal{K}A^2}\sin^2(\omega_0 t + \phi) + \frac{1}{2}\mathcal{K}A^2\cos^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}\mathcal{K}A^2\left(\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)\right)$$

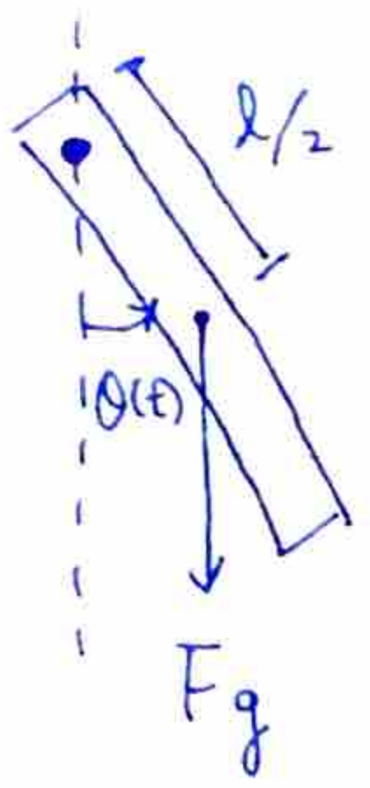
$$= \frac{1}{2}\cdot \mathcal{K}A^2$$ Constant !!!

↳ proportional to A^2 amplitude

↳ proportional to $\mathcal{K}$ "spring constant"



Slide 3



Let's look at this example:

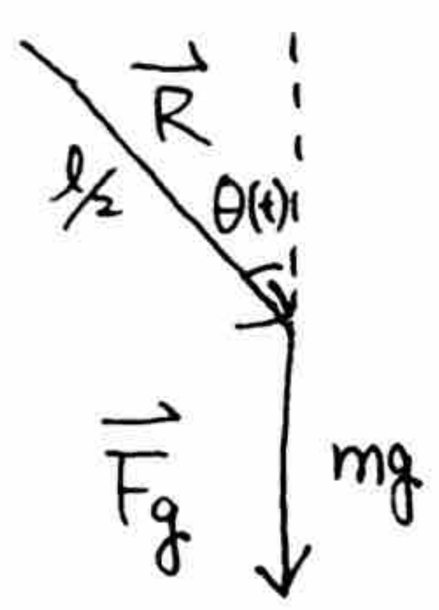
Newton's law $\vec{\tau} = I\vec{\alpha}$

Origin : $\theta = 0 \Rightarrow$ Pointing downward

Define Anti-clockwise Rotation to be Positive

Initial Condition : At $t=0 \Rightarrow \begin{cases} \theta(0) = \theta_{INITIAL} \\ \dot{\theta}(0) = 0 \end{cases}$

Force diagram:



$\vec{\tau} = \vec{R} \times \vec{F}$ since the system is rotating on a single plane $\Rightarrow$ drop the vector sign

$\tau = -mg\frac{l}{2}\sin\theta(t)$

Newton's law: $\tau = I\alpha(t) = I\ddot{\theta}(t) = \frac{-mgl}{2}\sin\theta(t)$

$\ddot{\theta}(t) = \frac{-mgl}{2I}\sin\theta(t)$ $\quad I = \frac{1}{3}ml^2$

$\Rightarrow \ddot{\theta}(t) = \frac{-3g}{2l}\sin\theta(t) = -\omega_0^2\sin\theta(t)$

$\omega_0 \equiv \sqrt{\frac{3g}{2l}}$

Now again: We have translated the physical situation to mathematics. This contains everything we know

We need to solve this equation

However, life is hard!

We don't know how to solve $\ddot{\theta} = -\omega_0^2\sin\theta$

Not the end of world, we can solve it by a computer

or .....

We can consider a special case : Small angle limit

$$\theta(t) \to 0 \quad \Rightarrow \quad \sin\theta(t) \approx x$$

$$\theta = 1^\circ \quad \Rightarrow \quad \frac{\sin\theta}{\theta} = 99.99\%$$

$$5^\circ \quad \Rightarrow \quad 99.9\%$$

$$10^\circ \quad 99.5\%$$

The approximation is quite good !

Then the equation of motion becomes:

$$\ddot{\theta}(t) = -\omega_0^2\,\theta(t) \qquad \omega_0 = \sqrt{\frac{3g}{2\ell}}$$

We have solved this in previous lectures!

$$\theta(t) = A\cos(\omega_0 t + \phi)$$

Initial conditions:

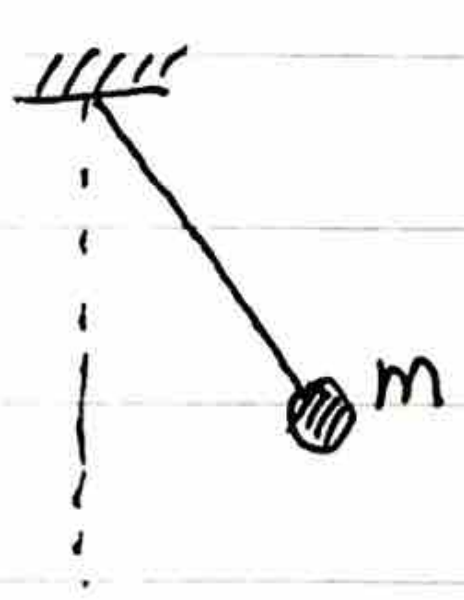
We conclude $0 = -\omega_0 A \sin\phi \Rightarrow \phi = 0$

$\theta_{INITIAL} = A$

$$\Rightarrow \quad \theta(t) = \theta_{INITIAL}\cos(\omega_0 t)$$

$$\omega_0 = \sqrt{\frac{3g}{2\ell}}$$

In case if you have not noticed:



$$\omega_0 = \sqrt{\frac{g}{\ell}}$$

SL3

All those systems

Now we will add a drag force :

$$\tau_{DRAG}(t) = -b\,\dot{\theta}(t)$$

2 $\sin\theta \approx \theta$ small oscillation

We choose this form : not because it is the most realistic description, but because this is solvable.
(also not bad at all !)

If we choose another form of drag force
⇒ have to solve it by computer

New teaching physics ⇒ That's why we use those approximation + assumption in class.

* EQUATION OF MOTION:

$$\ddot{\theta}(t) = \frac{\tau(t)}{I} = \frac{\tau_g(t) + \tau_{DRAG}(t)}{I}$$

$\theta(t)$ ss

$$= \frac{-mg\frac{l}{2}\sin\theta(t) - b\dot{\theta}(t)}{\frac{1}{3}ml^2}$$

small angle

$$\approx -\frac{3g}{2l}\theta(t) - \frac{3b}{ml^2}\dot{\theta}(t)$$

Define $\omega_0^2 = \frac{3g}{2l}$ $\quad \Gamma = \frac{3b}{ml^2}$

Again: The reason we define ω_0 and Γ is to simplify things, to make our life easier.

$$\Rightarrow \ddot{\theta}(t) + \Gamma\dot{\theta}(t) + \omega_0^2\theta(t) = 0$$

Oscillation Frequency

Question:
$\omega > \omega_0$ or $\omega \leq \omega_0$

Ans:

Now we want to solve this equation.

$$\ddot{\Theta}(t) + \Gamma\dot{\Theta}(t) + \omega_0^2\Theta(t) = 0$$

Use complex notation!

$$\Theta(t) = \mathrm{Re}\,(z(t)) \qquad z(t) = e^{i\alpha t}$$

$$\Rightarrow \quad \ddot{z}(t) + \Gamma\dot{z}(t) + \omega_0^2 z(t) = 0$$

$$(-\alpha^2 + i\Gamma\alpha + \omega_0^2)\, e^{i\alpha t} = 0$$

$e^{i\alpha t}$ is never 0.

$$\Rightarrow \quad \alpha^2 - i\Gamma\alpha - \omega_0^2 = 0$$

$$\Rightarrow \quad \alpha = \frac{i\Gamma \pm \sqrt{4\omega_0^2 - \Gamma^2}}{2} = \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$$

1. If $\omega_0^2 > \frac{\Gamma^2}{4}$ "Underdamped Oscillators"

The drag force is small

$\Rightarrow$ Define $\omega^2 \equiv \omega_0^2 - \frac{\Gamma^2}{4}$

$$z_+(t) = e^{-\frac{\Gamma}{2}t}\, e^{i\omega t}$$

$$z_-(t) = e^{-\frac{\Gamma}{2}t}\, e^{-i\omega t}$$

Ans: slower

$$\Theta_1(t) = \frac{1}{2}\left(Z_+(t) + Z_-(t)\right)$$

$$= e^{-\frac{\Gamma}{2}t} \cos\omega t$$

$$\Theta_2(t) = \frac{-i}{2}\left(Z_+(t) - Z_-(t)\right)$$

$$= e^{-\frac{\Gamma}{2}t} \sin\omega t$$

$$\Theta(t) = e^{-\frac{\Gamma}{2}t}\left[a \cos\omega t + b \sin\omega t\right]$$

or

$$\Theta(t) = Ae^{-\frac{\Gamma}{2}t}\left[\cos(\omega t + \phi)\right]$$

Use the initial condition :

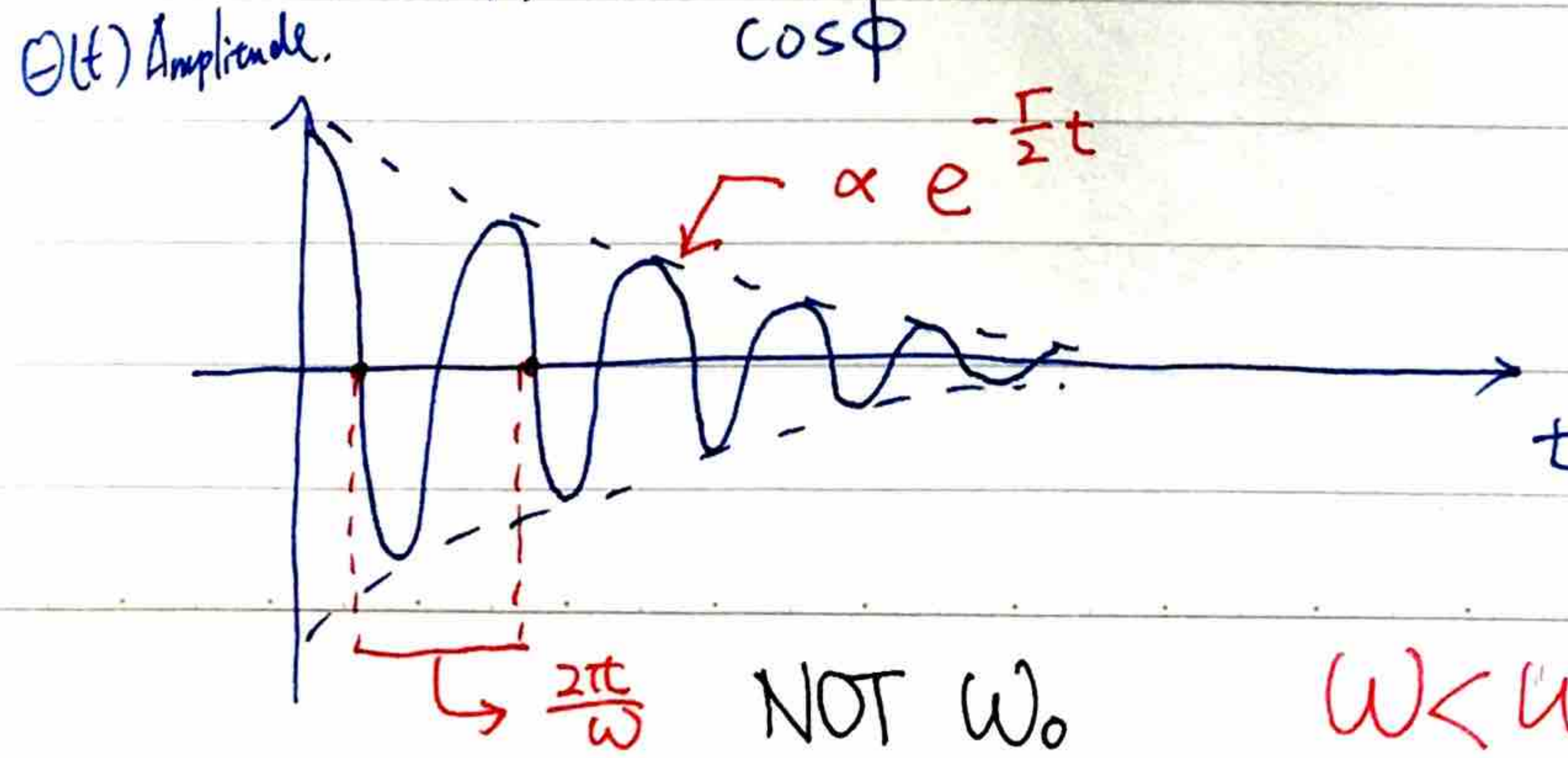
$$\Theta(0) = \Theta_{INITIAL} = A\cos\phi$$

$$\dot{\Theta}(0) = -\frac{A\Gamma}{2}\cos\phi - A\omega\sin\phi = 0$$

We can solve A and ϕ

$$\tan\phi = \frac{-\Gamma}{2\omega} \qquad \phi = \tan^{-1}\frac{-\Gamma}{2\omega}$$

$$A = \frac{\Theta_{INITIAL}}{\cos\phi}$$



NOT ω_0

$\omega < \omega_0$

Demo

2. $\omega_0^2 = \frac{\Gamma^2}{4}$ "Critically Damped Oscillator"

This means that $\omega = 0$!

Starting from 1.

$$\theta_1(t) = e^{-\frac{\Gamma}{2}t}\cos\omega t \xrightarrow{\omega\to 0} e^{-\frac{\Gamma}{2}t}$$

$$\theta_2(t) = e^{-\frac{\Gamma}{2}t}\sin\omega t \xrightarrow{\omega\to 0} 0$$

Not helpful

So instead.... we do:

$$\frac{\theta_2(t)}{\omega} = \frac{1}{\omega}e^{-\frac{\Gamma}{2}t}\sin\omega t \xrightarrow{\omega\to 0} t\,e^{-\frac{\Gamma}{2}t}$$

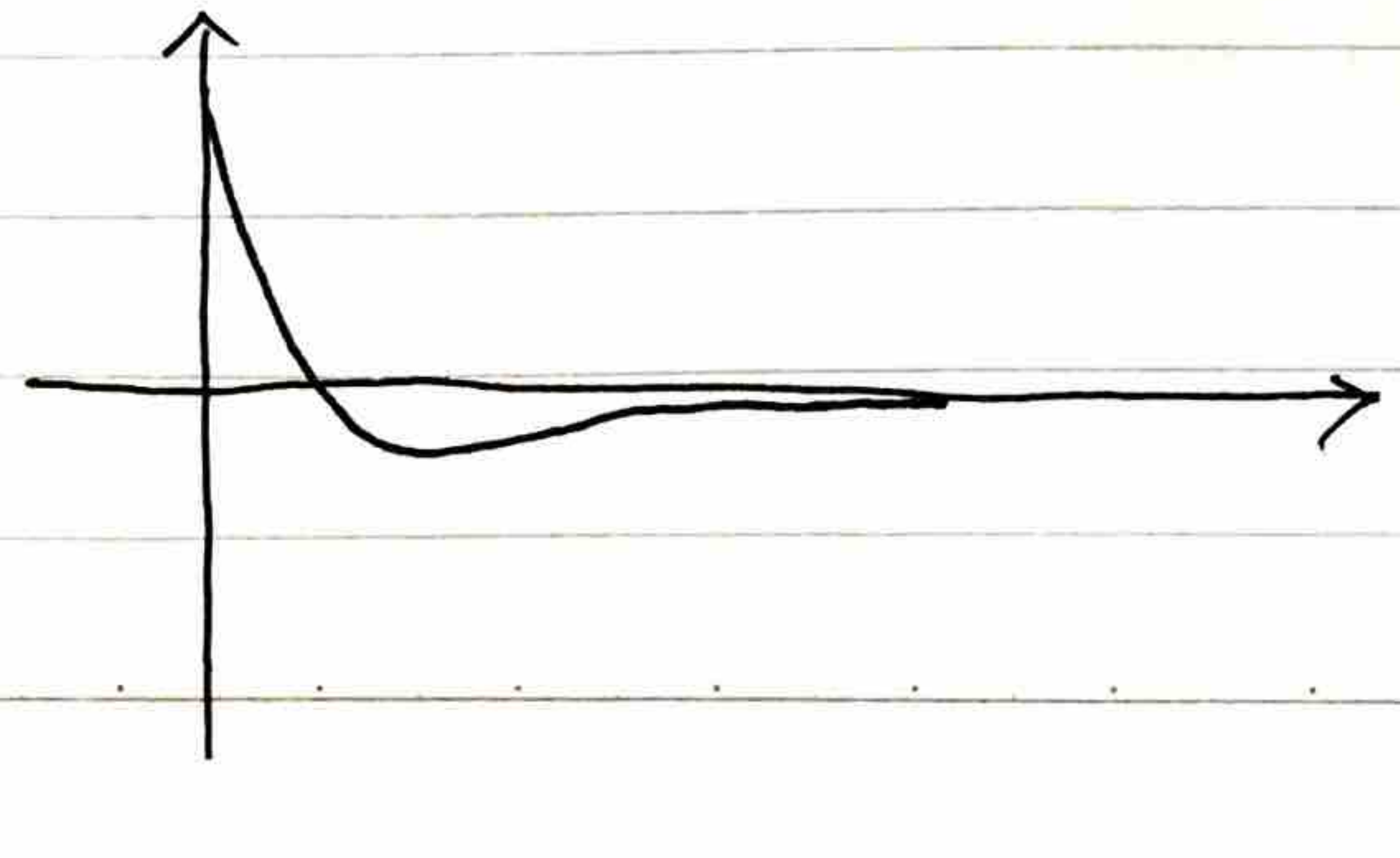
$\Rightarrow$ Linear combination of $\theta_1(t)$ and $\frac{\theta_2(t)}{\omega}$:

$$\theta(t) = (A + Bt)\,e^{-\frac{\Gamma}{2}t}$$

Prediction: No oscillation!

Application: ✓ Door close

✓ Suspension System



3. $\omega_0^2 < \frac{\Gamma^2}{4}$ Huge drag force !

$$\alpha = \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$$

"Overdamped Oscillator"

$$= i\left(\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)$$

Define $\Gamma_\pm = \frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$

$\Rightarrow$ Solution: $\Theta(t) = A_+ e^{-\Gamma_+ t} + A_- e^{-\Gamma_- t}$

No oscillation! Two exponentially decaying terms.