Massachusetts Institute of Technology

Physics 8.03

Practice Exam 1

Instructions

Please write your solutions in the white booklets. We will not grade anything written on the exam copy. This exam is closed book. No electronic equipment is allowed. All phones, tablets, computers etc. must be switched off.

Formula Sheet Exam 1

Springs and masses:

$$m\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + kx(t) = F(t)$$

More general differential equation with harmonic driving force:

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_d t)$$

Complex steady state solution

$$z_s(t) = z_0 e^{-i\omega_d t} \quad z_0 = Ae^{i\delta} = c + id \quad A = \sqrt{c^2 + d^2}$$

$$\delta = \arctan(d/c) \text{ for } c > 0 \text{ and } \delta = \arctan(d/c) + \pi \text{ for } c < 0$$

Physical system follows the real part of this solution

$$x_s(t) = Re(z_s(t)) \quad Re(z) = (z + z^*)/2$$

General solutions, including free oscillations:
For $\Gamma < \omega_0/2$ (under damped system):

$$x(t) = Re^{-\frac{\Gamma}{2}t}\cos\left(\sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}\; t + \theta\right) + x_s(t)$$

For $\Gamma = \omega_0/2$ (critically damped system):

$$x(t) = (R_1 + R_2 t)e^{-\frac{\Gamma}{2}t} + x_s(t)$$

For $\Gamma > \omega_0/2$ (over damped system):

$$x(t) = R_1 e^{-\left(\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + R_2 e^{-\left(\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + x_s(t)$$

Coupled oscillators

$$F_j = -\sum_{k=1}^{n} K_{jk} x_k$$

Examples for $n = 2$

$$\mathcal{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \quad \mathcal{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Matrix equation of motion, matrices $\mathcal{M}$, $\mathcal{K}$, $\mathcal{I}$ are $n \times n$, vectors $\mathcal{X}$, $\mathcal{Z}$ are $n \times 1$.

$$\frac{d^2}{dt^2}\mathcal{X}(t) = -\mathcal{M}^{-1}\mathcal{K}\mathcal{X}(t)$$

$$\mathcal{Z}(t) = \mathcal{A}e^{-i\omega t}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I})\mathcal{A} = 0$$

To obtain the frequencies of normal modes solve:

$$det(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I}) = 0$$

For $n = 2$

$$det\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M_{11}M_{22} - M_{12}M_{21}$$

If the system is driven by force one can find the response amplitudes $\mathcal{C}(\omega_d)$

$$\mathcal{F}(t) = \mathcal{F}_0 e^{-i\omega_d t}$$

$$\mathcal{W}(t) = \mathcal{C}(\omega_d)e^{-i\omega_d t}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega_d^2\mathcal{I})\mathcal{C}(\omega_d) = \mathcal{F}_0$$

Kirchoff's Laws (be careful about the signs!)

$$\text{Node}: \sum_i I_i = 0 \ \text{Loop}: \sum_i \Delta V_i = 0$$

$$\text{Capacitors}: \Delta V = \frac{Q}{C} \quad \text{Inductors}: \Delta V = -L\frac{dI}{dt} \quad \text{Current}: I = \frac{dQ}{dt}$$

Problem 1 (30 pts)

Consider mass M hanging on a massless vertical spring with spring constant k (see Figure 1). The mass is at rest. At $t = 0$ a piece of the mass falls off, leaving only a fraction α of the original mass attached to the spring. Gravitational acceleration is g. Assume that the mass moves along vertical y-axis. At $t = 0$ the mass was at $y = 0$.

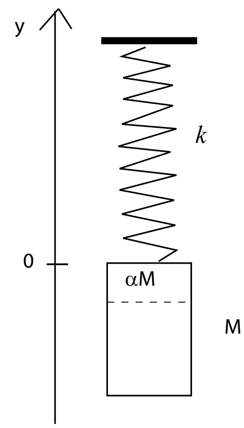

Figure 1: Broken Mass

a. Find the new equilibrium position as a function of the given parameters.
b. The mass will oscillate. What is the period of oscillations?
c. What is the time dependence of the vertical position, $y(t)$?
d. What is the amplitude and phase of the motion in terms of given parameters?
e. What is the kinetic and potential energy of mass m as a function of time?

Problem 2 (30 pts)

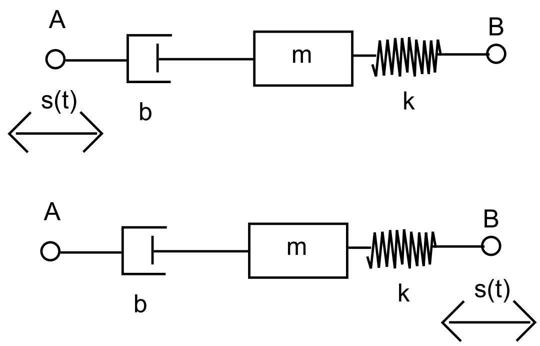

Figure 2: Two ways to drive an oscillator

Consider a simple oscillator with damping. Mass m is attached to a spring of spring constant k and a damper with damping force proportional to $-bv$. The spring and the damper are attached to the walls on the opposite sides of the mass (see Figure 2). The oscillator can be driven either by moving an attachment point on the damper (A) or the end of the spring (B). In both cases the position of the attachment point as a function of time is $s(t) = s_0 \cos(\omega_d t)$. For BOTH of the above cases, A and B, answer each of the following questions.

a. Write the equations of motion of the mass m.

b. Find the amplitude of steady state solution in terms of given parameters. Sketch the oscillation amplitude as a function of frequency ω_d.

c. What is the frequency ω_d at which the amplitude is at a maximum?

d. What is the behavior of the amplitude at $\omega_d \ll \omega_0$ and $\omega_d \gg \omega_0$?

Problem 3 (40 pts)

Consider the five different coupled oscillating systems in Figure 3 A-E. For each system please write a set of equations of motion for small amplitude harmonic oscillations. Define clearly the coordinates that you are using to describe each system. The equations should be written in matrix form: $(\mathcal{A} - \omega^2 \mathcal{I})C = 0$ where, for the familiar systems consisting of masses and springs, $\mathcal{A} = \mathcal{M}^{-1}\mathcal{K}$. There is also an equivalent matrix for LC circuits. DO NOT find the solutions to these equations!

Notes for the systems:

A. The ends of the springs which are not attached to a mass are fixed.

B. Ignore vertical motion of the masses.

C. Ignore effects due to the curvature of the ring.

D. Write the equations of motion in terms of the currents through each of the three inductors.

E. Consider only small-angle oscillations.

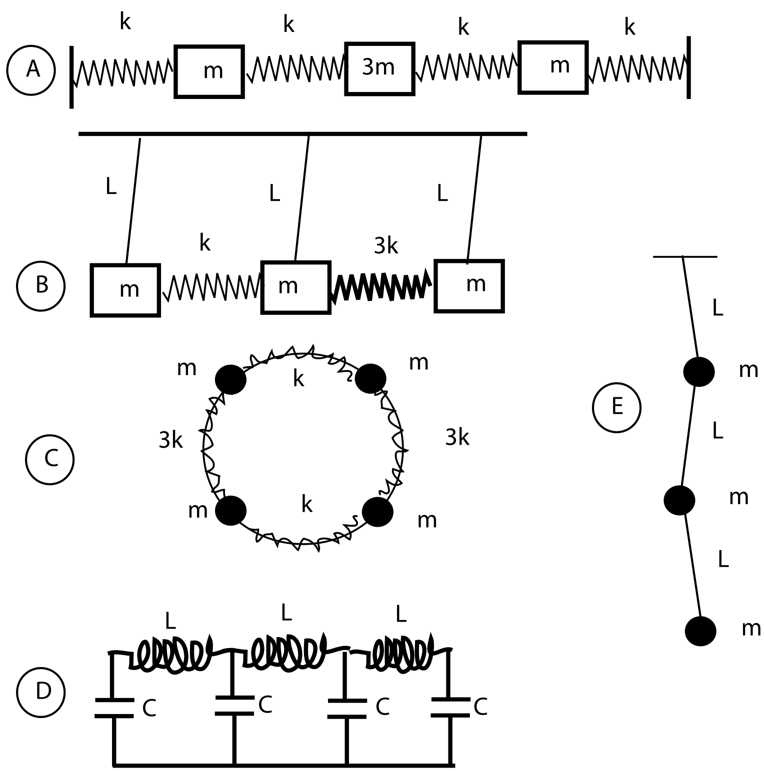

Figure 3: Five Oscillators