MASSACHUSETTS INSTITUTE OF TECHNOLOGY
PHYSICS DEPARTMENT

Physics 8.03: Vibrations and Waves

Practice Final **Exam 1 Solutions**

Problem 1: 16 Points

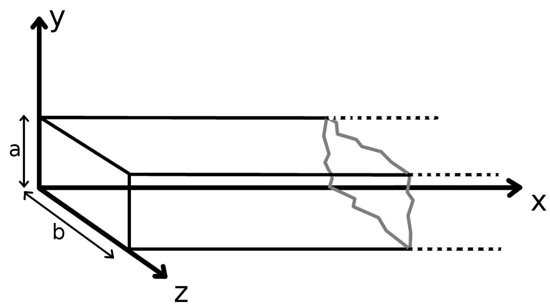


Figure 1: A perfectly conducting waveguide.

The electric field for a TE mode in an infinitely long (in the x direction) perfectly conducting rectangular waveguide ($a < b$) is given by;

$$\vec{E}(x, y, z, t) = E_0 \cos(k_y y + \phi_y) \cos(k_x x - \omega t)\hat{z} \tag{1}$$

(1.a)

(4pts) Find k_y and ϕ_y that satisfy the boundary conditions.

We need to have $E_{\parallel} \to 0$ at the boundary with the conductor. The electric field is in the $\hat{z}$ direction, so we need $E = 0$ at (i) $y = 0$ and at (ii) $y = b$; (i) implies that $\phi_y = \pm\frac{\pi}{2}$, so $E \propto \sin(k_y y)$. Condition (ii) implies that $k_y = \frac{n\pi}{a}$, where n is some integer greater than zero. So we may rewrite the electric field as;

$$E(x, y, z, t) = E_0 \sin\left(\frac{n\pi}{a} y\right) \cos\left(k_x x - \omega t\right)\hat{z} \tag{2}$$

(1.a)

(1.b)

(4pts) Write down the dispersion relation for this mode of the waveguide.

We can obtain the dispersion relation from the wave equation;

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{3}$$

Putting the solution for $\vec{E}$ into the wave equation we find;

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 \tag{4}$$

$$\frac{\omega^2}{c^2} = k_x^2 + \left(\frac{n\pi}{a}\right)^2 \tag{5}$$

$$\Rightarrow k_x = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{n\pi}{a}\right)^2} \tag{6}$$

(1.b)

(1.c)

(4pts) What is the lowest frequency that will propagate in this mode?

The lowest frequency that will propagate occurs at $n = 1$, in this case the dipersion relation becomes;

$$k_x = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{\pi}{a}\right)^2} \tag{7}$$

so the lowest frequency that results in a non-imaginary wave number is;

(1.c)

$$\omega_{\text{cutoff}} = \frac{\pi c}{a} \tag{8}$$

$$\Rightarrow f_{\text{cutoff}} = \frac{c}{2a} \tag{9}$$

(1.d)

(4pts) What is the magnetic field $\vec{B}(x, y, z, t)$ associated with the electric field of this mode?

We can find the magnetic field associated with this mode by the use of Faraday's law;

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{10}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} = -\frac{\partial \vec{B}}{\partial t} \tag{11}$$

(1.d)

$$E_0 k_y \cos(k_y y) \cos(k_x x - \omega t)\hat{x} + E_0 \sin(k_y y) k_x \sin(k_x x - \omega t)\hat{y} = -\frac{\partial \vec{B}}{\partial t} \tag{12}$$

$$\vec{B}(x, y, z, t) = E_0 \left(\frac{k_y}{\omega}\right) \cos(k_y y) \sin(k_x x - \omega t)\hat{x} - E_0 \left(\frac{k_x}{\omega}\right) \sin(k_y y) \cos(k_x x - \omega t)\hat{y} \tag{13}$$

Problem 2: 16 Points

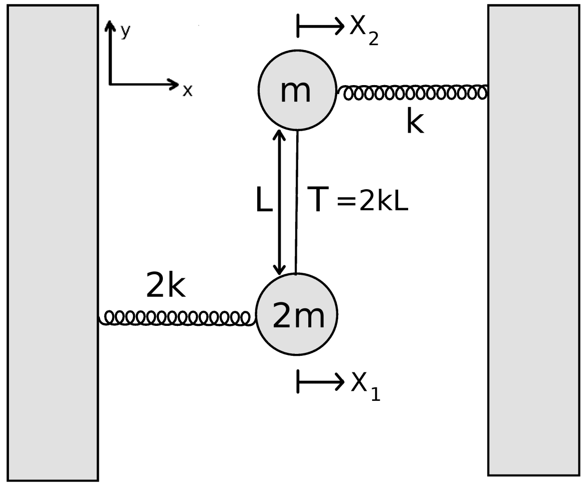


Figure 2: A system of coupled oscillators.

The figure above shows a system of masses. The mass of $2m$ is connected to an immobile wall with a spring of constant $2k$, while the mass of m is connected to an immobile wall with a spring of constant k. The masses are then coupled to each other with an elastic band of length L, under tension $T = 2kL$. The masses are constrained to move in the x direction only. At equilbrium the masses have the same x position and the springs are uncompressed. There is no friction or gravity. The displacements from equilibrium are small enough ($x_1, x_2 \ll L$), so that the tension in the band stays constant.

(2.a)

(5pts) Write down the coupled differential equations describing the displacement of the masses from equilibrium $\{x_1, x_2\}$.

For x_1 we have from Newton's law;

$$2m\ddot{x}_1 = -2kx_1 - T\sin\theta \tag{14}$$

Where $\sin\theta = (x_1 - x_2)/L$. Putting this in, we find;

$$2m\ddot{x}_1 = \left(-2k - \frac{T}{L}\right)x_1 + \frac{T}{L}x_2 \tag{15}$$

$$\ddot{x}_1 = \left(-\frac{k}{m} - \frac{T}{2mL}\right)x_1 + \frac{T}{2mL}x_2 \tag{16}$$

$$\ddot{x}_1 = \frac{-2k}{m}x_1 + \frac{k}{m}x_2 \tag{17}$$

While for x_2 we find; (2.a)

$$m\ddot{x}_2 = -kx_2 + T\sin\theta \tag{18}$$

$$m\ddot{x}_2 = \left(-k - \frac{T}{L}\right)x_2 + \frac{T}{L}x_1 \tag{19}$$

$$\ddot{x}_2 = \frac{2k}{m}x_1 - \frac{3k}{m}x_2 \tag{20}$$

Now calling $\omega_0^2 = k/m$ we can rewrite these as;

$$\ddot{x}_1 = -2\omega_0^2 x_1 + \omega_0^2 x_2 \tag{21}$$

$$\ddot{x}_2 = 2\omega_0^2 x_1 - 3\omega_0^2 x_2 \tag{22}$$

(2.b)

(7pts) Find the normal mode frequencies of the system.

In matrix form we can rewrite the equations of motion (using the ansatz $\ddot{x}_i = -\omega^2\ddot{x}_i$);

$$\begin{bmatrix} -2\omega_0^2 & \omega_0^2 \\ 2\omega_0^2 & -3\omega_0^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\omega^2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{23}$$

And solving for the possible values of ω we have;

$$\begin{bmatrix} -2\omega_0^2 + \omega^2 & \omega_0^2 \\ 2\omega_0^2 & -3\omega_0^2 + \omega^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{24}$$

$$\left(-2\omega_0^2 + \omega^2\right)\left(-3\omega_0^2 + \omega^2\right) - 2\omega_0^4 = 0 \tag{25}$$

(2.b)

$$6\omega_0^4 - 5\omega_0^2\omega^2 + \omega^4 - 2\omega_0^4 = 0 \tag{26}$$

$$\left(4\omega_0^2 - \omega^2\right)\left(\omega_0^2 - \omega^2\right) = 0 \tag{27}$$

So the two possible values of ω are;

$$\omega_1 = \omega_0 = \sqrt{\frac{k}{m}} \tag{28}$$

and

$$\omega_2 = 2\omega_0 = 2\sqrt{\frac{k}{m}} \tag{29}$$

(2.c)

(4pts) On the two figures included on the next page sketch the normal modes of the system, be sure to clearly indicate both the magnitude and direction of the motion of the masses.

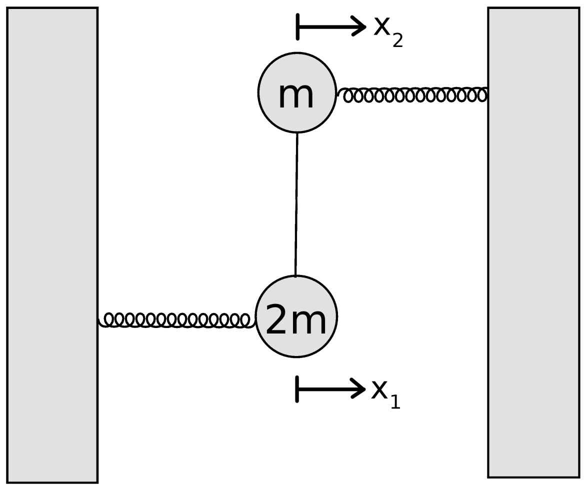
In mode 1 (ω_1), shown in the first figure the ratio of the amplitudes of the motion of the masses is;

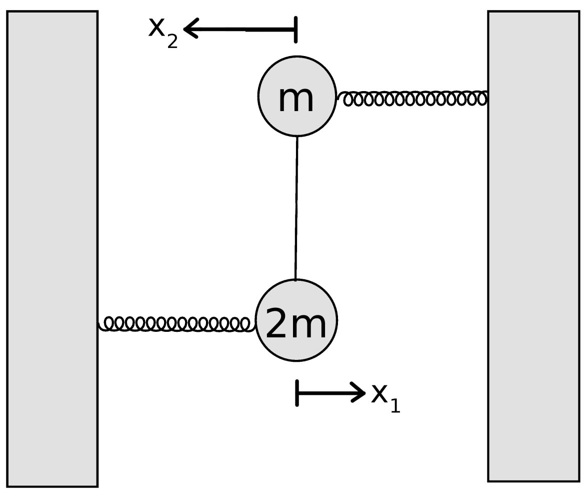
$$x_2/x_1 = 1 \tag{30}$$

While in mode 2 (ω_2), shown in the second figure, the ratio of the amplitudes of the motion of the masses is;

$$x_2/x_1 = -2 \tag{31}$$

(2.c)





Problem 3: 16 Points

(3.a)

(5pts) An optical fiber consists of a solid rod of material with index of refraction n_f surrounded by a cylindrical shell of material with index n_c. Find the largest angle θ so that a wave incident on the solid rod from air with index n_a remains in the solid rod (express your answer in terms of n_f, n_c, and n_a).

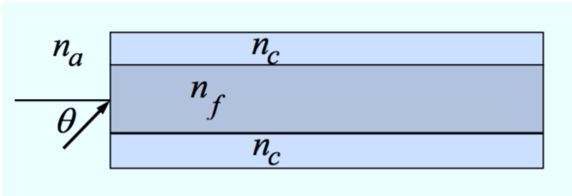


Figure 3: An optical fiber.

All rays in the solid rod which are incident on the cylindrical shell at an angle less than the critical angle $\theta_c = \sin^{-1}(n_c/n_f)$ will be trapped in the rod. So the largest angle θ that can be incident on the rod and remain inside is given by;

$$n_a \sin\theta = n_f \sin\theta_2 \tag{32}$$

$$\theta_2 = \frac{\pi}{2} - \theta_c \tag{33}$$

$$\Rightarrow \sin\theta = \frac{n_f}{n_a} \sin\left(\frac{\pi}{2} - \theta_c\right) \tag{34}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{n_f}{n_a} \sin\left(\frac{\pi}{2} - \theta_c\right)\right) \tag{35}$$

(3.a)

Or equivalently, we can write;

$$\frac{n_a}{n_f} \sin\theta = \sin\theta_2 = \frac{\sqrt{n_f^2 - n_c^2}}{n_f} \tag{36}$$

$$\sin\theta = \left(\frac{n_f}{n_a}\right)\left(\frac{\sqrt{n_f^2 - n_c^2}}{n_f}\right) \tag{37}$$

$$\theta = \sin^{-1}\left(\frac{1}{n_a}\sqrt{n_f^2 - n_c^2}\right) \tag{38}$$

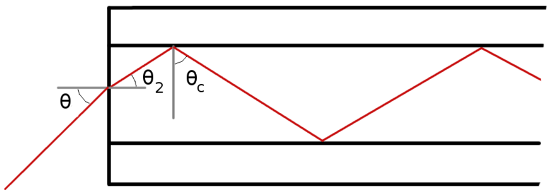


Figure 4: Path of an extreme ray.

(3.b)

(4pts) Unpolarized light propagating in vacuum reflects off the surface of a liquid with index n. The reflected ray strikes a screen 25cm away at a height of 20cm and is observed to be 100% polarized. What is is n?

If the reflected light is 100% polarized then it must be incident at Brewster's angle. So we have;

$$\theta_B = \tan^{-1}(n_2/n_1) = \tan^{-1}(n_2) = \frac{\pi}{2} - \tan^{-1}(20/25) \tag{39}$$

$$\Rightarrow n_2 = \tan\left(\frac{\pi}{2} - \tan^{-1}(4/5)\right) \tag{40}$$

Or more simply, we can write;

(3.b)

$$\tan^{-1}(n_2/n_1) = \theta_B \tag{41}$$

$$\tan\theta_B = 25/20 = 5/4 \tag{42}$$

$$\Rightarrow \tan(\tan^{-1}(n_2/n_1)) = 5/4 \tag{43}$$

$$\Rightarrow n_2 = 5/4 \tag{44}$$

(3.c)

(7pts) Consider a medium in which waves propagate with a dispersion relation

$$\omega^2 = \omega_0^2 + A^2k^2 \tag{45}$$

where ω is the wave (angular) frequency, k is the wave number, and ω_0 and A are real constants.

(i) What is the range of frequencies ω for which waves can propagate?

(ii) Compute v_{phase} and v_{group}. Make a carefully labeled sketch of each as a function of ω in the plots below.

(i) Rearranging the dispersion relation we find that the wave number in terms of the frequency is given by;

$$k = \frac{1}{A}\sqrt{\omega_0^2 + A^2k^2} \tag{46}$$

So the lowest frequency that results in a non-imaginary wave number is $\omega = \omega_0$, so the range of frequencies that can propagate are $\omega_0 < \omega < \infty$.

(ii) The phase velocity is given by ω/k, so we find;

(3.c)

$$v_p = \omega/k = \frac{\omega}{\frac{1}{A}\sqrt{\omega^2 - \omega_0^2}} = \frac{A\omega}{\sqrt{\omega^2 - \omega_0^2}} \tag{47}$$

While the group velocity is given by $\frac{d\omega}{dk}$;

$$v_g = \frac{d\omega}{dk} = A^2\frac{k}{\omega} = A\sqrt{1 - \left(\frac{\omega_0}{\omega}\right)^2} \tag{48}$$

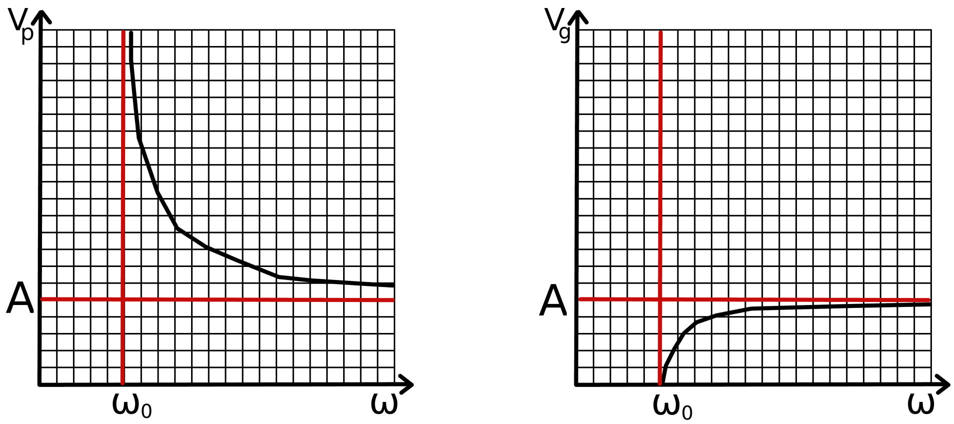


Figure 5: The phase/group velocity

Problem 4: 16 Points

A monochromatic beam is incident on N slits, which results in a intensity pattern as a function of angle on a screen some distance away as shown in the figure below. Each slit has a width D and the distance between the centers of the slits is d. The distance between the screen and the slits is very large.

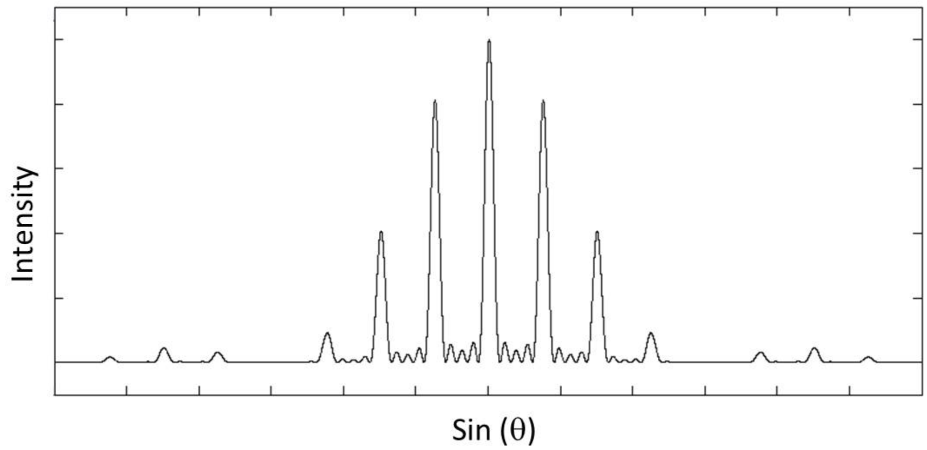


Figure 6: Interference pattern due to N slits.

From the pattern deduce the following:

(4.a)

(6pts) The number of slits N on which the beam is incident. Explain your reasoning.

There are three small bumps between peaks. Therefore, $N = 5$ (Note that there will be $N - 2$ small bumps for an N slit interference pattern). (4.a)

(4.b)

(6pts) The ratio d/D. Explain your reasoning.

First, recall that interference peaks appear at $\frac{n\lambda}{d}$, while diffraction minima appear at $\frac{m\lambda}{D}$ where n and m are integers. From the interference pattern above, we can see 0th, 1st, 2nd, and 3rd peaks, but cannot see the 4th peak. This means that 4th peak is cancelled due to the diffraction minima. So we notice that the 1st diffraction minima and 4th interference peak are at the same position. Therefore, $\frac{4\lambda}{d} = \frac{\lambda}{D}$, and therefore $d/D = 4$. (4.b)

(4.c)

(4pts) Now suppose that the width of the slits is reduced to ~ 0, while the intensity of the monochromatic beam is increased so that the intensity of the central maximum is unchanged. On top of the plot (showing the original intensity pattern in dashed lines) on the next page, draw the resulting intensity pattern.

We recall from the demonstration in class that as the size of a slit is reduced the diffraction pattern is broaded, in the limit that the width of the slits goes to zero, the diffraction pattern is infinitely broadened so that it longer modulates the interference pattern. (4.c)

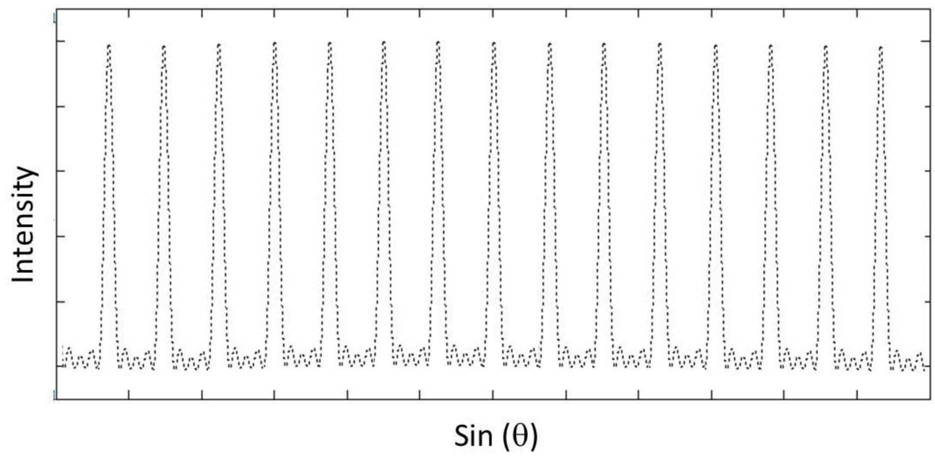


Figure 7: The resulting intensity pattern as $D \to 0$.

Problem 5: 20 Points

A string of length $2L$ with mass density μ is placed under tension T and is fixed at both ends. At time $t = 0$, the displacement of the string is zero everywhere but it is struck so that a transverse velocity is imparted to a section of the string. The intial conditions of the string are ($a \ll L$);

$$y(x, t = 0) = 0 \tag{49}$$

$$\dot{y}(x, t = 0) = \begin{cases} v_0 & : \quad L - a \leq x < L \\ -v_0 & : \quad L \leq x < L + a \\ 0 & : \quad \text{elsewhere} \end{cases} \tag{50}$$

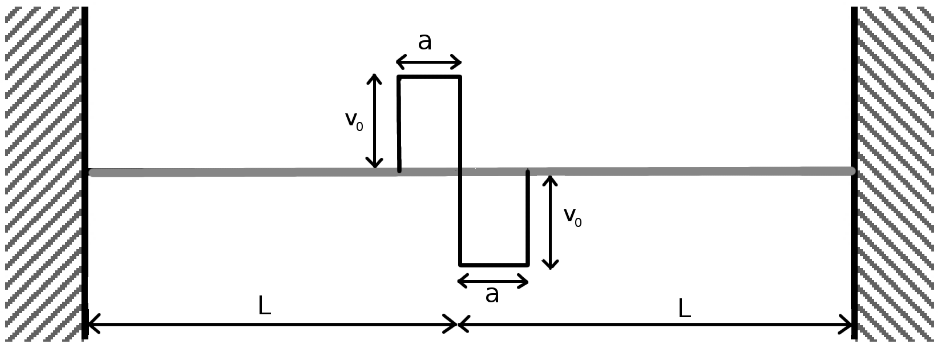


Figure 8: The intial transverse velocity of the string at time $t = 0$. The initial displacement is zero everywhere.

(5.a)

(3pts) Using the plot (Figure #9) provided on the next page, sketch the first three normal modes of vibration for this string, regardless of whether or not they are excited.

(5.b)

(10pts) What is the amplitude of the n-th normal mode after the string is struck? What is the lowest unexcited mode?

In this case, the displacement of the string can be expressed in terms of a sum of normal modes with the form;

$$y(x,t) = \sum_n A_n \sin(k_n x)\sin(\omega_n t + \cancelto{0}{\phi}) \tag{51}$$

with $\omega_n = k_n v = k_n\sqrt{\frac{T}{\mu}}$ and $k_n = \frac{n\pi}{2L}$.
So the transverse velocity (terms of a normal mode expansion) can be written;

$$\dot{y}(x,t) = \sum_n A_n \omega_n \sin(k_n x)\cos(\omega_n t) \tag{52}$$

And the amplitude of excitation on each normal mode A_n can be found by;

$$A_n\omega_n = \frac{2}{2L}\int_0^{2L} \dot{y}(x,t=0)\sin(k_n x)dx \tag{53}$$

$$A_n = \frac{1}{\omega_n L}\left[\int_{L-a}^{L} v_0\sin(k_n x)dx - \int_{L}^{L+a} v_0\sin(k_n x)dx\right] \tag{54}$$

(5.b)

$$A_n = -\frac{v_0}{\omega_n k_n L}\left[\cos(k_n x)|_{L-a}^{L} - \cos(k_n x)|_{L}^{L+a}\right] \tag{55}$$

$$A_n = -\frac{v_0}{\omega_n k_n L}\left[\cos(k_n L) - \cos(k_n(L-a)) - \cos(k_n(L+a)) + \cos(k_n L)\right] \tag{56}$$

$$A_n = -\frac{v_0}{\omega_n k_n L}\left[2\cos(k_n L) - 2\cos(k_n L)\cos(k_n a)\right] \tag{57}$$

$$A_n = -\frac{2v_0}{\omega_n k_n L}(1-\cos(k_n a))(cos(k_n L) \tag{58}$$

$$A_n = -\frac{v_0}{v}\frac{8L}{n^2\pi^2}\left[\left(1-\cos\left(\frac{n\pi}{2}\frac{a}{L}\right)\right)\cos\left(\frac{n\pi}{2}\right)\right] \tag{59}$$

$$A_n = -\frac{v_0}{\sqrt{\frac{T}{\mu}}}\frac{8L}{n^2\pi^2}\left[\left(1-\cos\left(\frac{n\pi}{2}\frac{a}{L}\right)\right)\cos\left(\frac{n\pi}{2}\right)\right] \tag{60}$$

From the last equation above we can see that the $n=1$ mode is unexcited. We could also see that this mode would be unexcited from symmetry, since the $n=1$ mode is symmetric about the center of the string, while the initial conditions are anti-symmetric. Similarly any mode which is symmetric about the center of the string will be unexcited.

(Problem continues on the next page.)

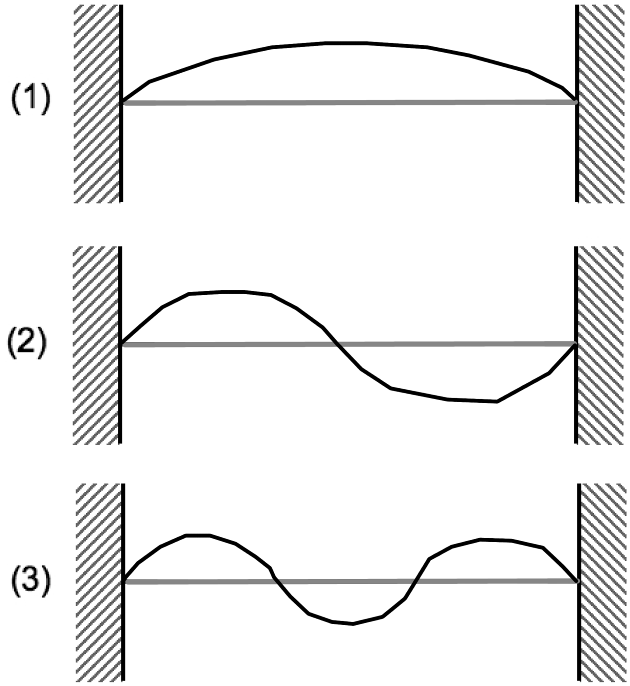


Figure 9: Plot the first three normal modes.

(5.c)

(5pts) Sketch the *displacement* of the string at time $t = \frac{L}{2}\sqrt{\frac{\mu}{T}}$ in the plot below.

The diplacement slits into two counter propagating waves, each with height $h = \frac{av_0}{\sqrt{T/\mu}}$, as shown below.

(5.c)

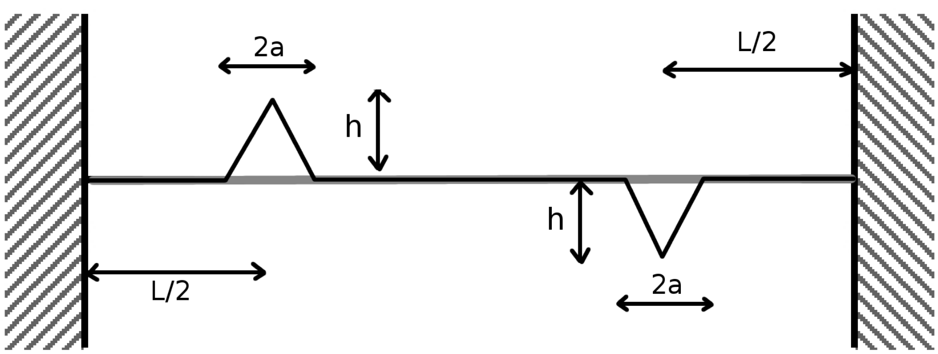


Figure 10: Plot the displacement of the string.

Problem 6: 18 Points

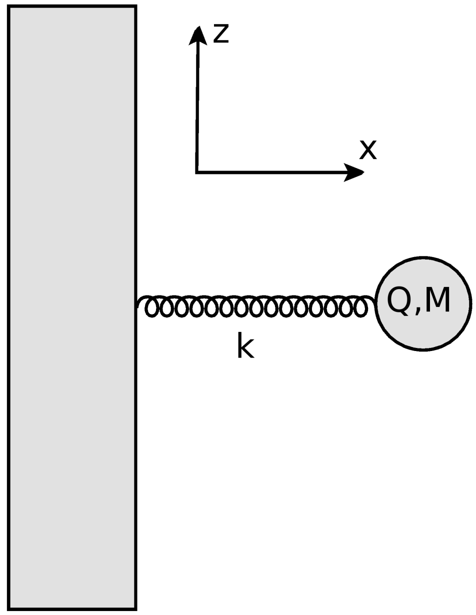


Figure 11: An oscillating charge.

A charged particle of mass M and charge $+Q$ is attached to the end of a spring of spring constant k. The spring lies along the x-axis and the equilibrium point is at the origin. The particle is displaced from equilibrium by a distance A in the x direction, and released at $t = 0$. Assume that the size of the particle is much smaller than A, so it can be treated as a point charge and that the damping rate is very small.

(6.a)

(4pts) Calculate the electric field radiated by the particle along an arbitrary direction in the $x - z$ plane, at a distance R, where $R \gg A$.

Assuming that the damping for the charge is neglectable, then the particle's motion is just that of a S.H.O, whose solution is;

$$x(t) = A\cos(\omega_0 t) \quad (61)$$

$$\ddot{x}(d) = -A\omega^2 \cos(\omega_0 t) \quad (62)$$

with $\omega_0 = \sqrt{k/m}$. Now the electric field is given by;

$$\vec{E}(\vec{R}, t) = -\frac{q\vec{a}_\perp(t')}{4\pi\epsilon_0 Rc^2} = -\frac{q(\hat{n} \times (\hat{n} \times \vec{a}(t')))}{4\pi\epsilon_0 Rc^2} \quad (63)$$

(6.a)

Where $\hat{n}$ is the unit vector pointing from the source to the observer. In the $x - z$ plane we have $\hat{n} = \sin\theta\hat{x} + \cos\theta\hat{z}$. So the electric field is given by;

$$\left(\frac{qA\omega_0^2}{4\pi\epsilon_0 c^2 R}\right)\cos(\omega_0(t - R/c))(\cos^2\theta\hat{x} - \cos\theta\sin\theta\hat{z}) \quad (64)$$

(6.b)

(4pts) Calculate the total time averaged power radiated by the particle.

The total power radiated by the particle is just given to us by the Larmor formula;

$$P(t) = \frac{q^2 a^2(t')}{6\pi\epsilon_0 c^3} = \frac{q^2 A^2 \omega_0^4 \cos^2(\omega_0(t - R/c))}{6\pi\epsilon_0 c^3} \tag{65}$$

(6.b)

Time averaging this results in;

$$\langle P \rangle = \frac{q^2 A^2 \omega_0^4}{12\pi\epsilon_0 c^3} \tag{66}$$

(6.c)

(6pts) Assuming that the power radiated does not change appreciably as a function of time, give a simple rough estimate of the time it will take for the particle to decrease its amplitude of oscillation to $1/e$ of its initial value. Is this assumption realistic?

With the assumption that the power radiated does not change much over time, then the time t it would take for the original amplitude of motion to decrease to $1/e$ of its value can by found by;

$$\frac{1}{2} m\omega_0^2 A^2 - \langle P \rangle t = \frac{1}{2} m\omega_0^2 \left(\frac{A}{e}\right)^2 \tag{67}$$

$$\Rightarrow t = \frac{m\omega_0^2 A^2}{2\langle P \rangle} \left(1 - \left(\frac{1}{e}\right)^2\right) \tag{68}$$

(6.c)

$$t \approx 0.86 \times \frac{6\pi m\epsilon_0 c^3}{q^2 \omega_0^2} \tag{69}$$

This assumption is not quite realistic since the power radiated by the charge is proportional to the amplitude of its motion squared and the amplitude of the motion is decreasing as energy is radiated away. This method will always underestimate the time needed for the amplitude to decrease

(6.d)

(4pts) A more refined estimate can be obtained using that $dA/dt = (dA/dE) \times (dE/dt)$, and using the average power radiated over a given cycle for dE/dt. Use this to calculate the time it will take the particle to decrease its oscillation amplitude to $1/e$ of its initial value.

The energy E is given by;

$$E = \frac{m\omega_0^2 A^2}{2} \tag{70}$$

so we have;

$$\frac{dA}{dE} = \frac{1}{m\omega_0^2 A} \tag{71}$$

Whereas the average power radiated per cycle is what we found in part (b);

$$\frac{dE}{dt} = \langle P \rangle = \frac{q^2 A^2 \omega_0^4}{12\pi\epsilon_0 c^3} \tag{72}$$

Combining these to find dA/dt we get;

$$\frac{dA}{dt} = \frac{dA}{dE}\frac{dE}{dt} = \left(\frac{1}{m\omega_0^2 A}\right)\left(\frac{q^2 A^2 \omega_0^4}{12\pi\epsilon_0 c^3}\right) = \frac{q^2 A \omega_0^2}{12\pi\epsilon_0 c^3 m} \tag{73}$$

(6.d)

Now integrating to find the time needed for the amplitude to decrease from A to A/e, we have;

$$\int_{A/e}^{A} \frac{dA}{A} = \left(\frac{q^2\omega_0^2}{12\pi\epsilon_0 c^3 m}\right)\int dt \tag{74}$$

$$\ln(A) - \ln(A/e) = \ln(e) = 1 = \left(\frac{q^2\omega_0^2}{12\pi\epsilon_0 c^3 m}\right) t \tag{75}$$

so the time t is given by;

$$t = \left(\frac{12\pi\epsilon_0 c^3 m}{q^2\omega_0^2}\right) \tag{76}$$