Welcome back to 8.033!

Image courtesy of Wikipedia.

Hendrik Antoon Lorentz,
Dutch, 1853-1928,
Nobel Prize 1902

Today: Deriving the Lorentz transformation

- How to transform between inertial frames
- What it all means
- **Key people:** Lorentz, Einstein

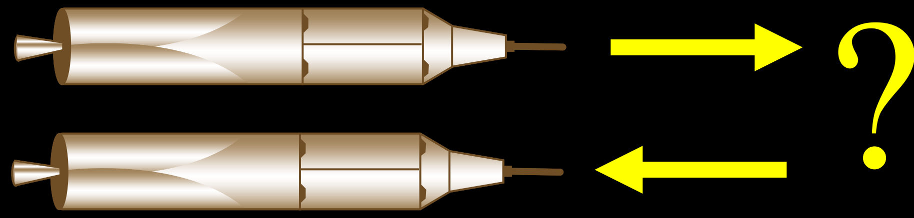

Figure by MIT OCW.

ANSWER:

- Translation:

$$\begin{cases} \mathbf{r}' = \mathbf{r} + \Delta r \\ t' = t + \Delta t \end{cases}$$

- Rotation:

$$\begin{cases} \mathbf{r}' = \mathbf{R}\mathbf{r} \\ t' = t \end{cases}$$

- Galilean:

$$\begin{cases} \mathbf{r}' = \mathbf{r} - \mathbf{v}t \\ t' = t \end{cases}$$

Battle Plan

We'll follow Einstein's approach and derive everything from two postulates:

1. The laws of physics are the same in all inertial frames.
2. The speed of light is same in all inertial frames.

Comments:

- 2 follows from 1 if we consider the speed of light one of the laws of physics.
- Einstein denoted inertial frame invariance "special relativity"
- As opposed to "general relativity", the generalization to non-inertial frames.

Inertial frames done carefully: rods & clocks

- Define coordinate system with three perpendicular rigid measuring rods
- Define time with *local* clocks
- Synchronize clocks with light pulses
- Minkowski diagram of synchronization procedure
- Minkowski diagram basics
- N.B. Don't confuse frame simultaneity with *seeing things* simultaneously: If you saw SN 1987A and a camera flash at the same time, did these two flashes go off simultaneously in your inertial frame?
- *The* time t in an inertial frame is also called *bookkeeper's time.* Don't confuse with the time when you *see* something happen.
- 1st shocker: Simultaneity is relative! Must abandon $t' = t$.

Derivation of the Lorentz transform

- Let's define 4-vectors that have units of length:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}.$$

- Given $\mathbf{v}$, the new 4-vector $\mathbf{x}'$ is some function of $\mathbf{x}$ - which function?
- Translational invariance implies linearity:

$$\mathbf{x}' = \mathbf{\Lambda}(\mathbf{v})\mathbf{x} + \mathbf{x}_0,$$

for some offset $\mathbf{x}_0$ and some Lorentz matrix

$$\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} \end{pmatrix}.$$

- Why? Because
$$\mathbf{x}_2' - \mathbf{x}_1' = \mathbf{\Lambda}(\mathbf{v})(\mathbf{x}_2 - \mathbf{x}_1)$$
for linear relation — for any nonlinear (rigorously, non-affine) relation, the difference $(\mathbf{x}_2' - \mathbf{x}_1')$ won't depend only on the difference $(\mathbf{x}_2 - \mathbf{x}_1)$.

- Notation warning: book uses notation where 4th coordinate is t, not ct, so there things get uglier and not all Λ-coefficients are dimensionless.

- Notation warning: book uses $\mathbf{a}$, we use $\mathbf{\Lambda}$ since it's more standard these days.

- Velocity sign convention: velocity of primed frame in unprimed frame is $\mathbf{v}$, so velocity of unprimed frame in primed frame is $-\mathbf{v}$

- WLOG no translation: $\mathbf{x}_0 = \mathbf{0}$ (we can always translate later), so simply need to find the 4×4 matrix $\mathbf{\Lambda}(\mathbf{v})$

- (WLOG=without loss of generality.)

- WLOG no rotation (we can always rotate later), so $\mathbf{v} = \mathbf{0}$ case gives identity matrix:

$$\boldsymbol{\Lambda}(\mathbf{0}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- WLOG $\mathbf{v}$ in x-direction, since we can always rotate to make it so

- Our transformation respects rotational symmetry around the x-axis, so neither x' nor t' can depend on y or z, *i.e.*, we have $\Lambda_{12} = \Lambda_{13} = \Lambda_{42} = \Lambda_{43} = 0$.

- The x-axis gets transformed into

$$\begin{pmatrix} \Lambda_{11} & 0 & 0 & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \\ \Lambda_{31} \\ \Lambda_{41} \end{pmatrix},$$

so we have $\Lambda_{21} = \Lambda_{31} = 0$ since, by construction, the spatial part of the x-axis coincides continuously with the x'-axis.

- Consider events with $x = t = 0$. They get transformed into

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & 0 & 0 & \Lambda_{14} \\ 0 & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ 0 & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} 0 \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \Lambda_{22} y + \Lambda_{23} z \\ \Lambda_{32} y + \Lambda_{33} z \\ 0 \end{pmatrix}.$$

  So all events in this two-dimensional (y, z)-plane are simultaneous in *both* frames (with $t' = t = 0$), making it trivial to compare measuring rods in the two frames since their two endpoints can coincide in space and time. This implies that the 2×2 transformation matrix in this plane must be the identity matrix, *i.e.*, $\Lambda_{23} = \Lambda_{32} = 0$ and $\Lambda_{22} = \Lambda_{33} = 1$.

- An object moving uniformly with $x = vt$ in the unprimed frame remains at rest at the origin in the primed frame, so

$$\begin{pmatrix} \Lambda_{11} & 0 & 0 & \Lambda_{14} \\ 0 & 1 & 0 & \Lambda_{24} \\ 0 & 0 & 1 & \Lambda_{34} \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} vt \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} \Lambda_{11}vt + \Lambda_{14}ct \\ \Lambda_{24}ct \\ \Lambda_{34}ct \\ \Lambda_{41}vt + \Lambda_{44}ct \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct' \end{pmatrix}$$

  so we have $\Lambda_{24} = \Lambda_{34} = 0$ and

$$\Lambda_{14} = -\beta\Lambda_{11},$$

  where we've defined

$$\beta \equiv \frac{v}{c}.$$

- (When we do research using relativity, we normally use units where $c = 1$, so that we can write simply $\beta = v$.)

- Progress update:

$$\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \Lambda_{11} & 0 & 0 & -\beta\Lambda_{11} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix}, \tag{1}$$

i.e.

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & 0 & 0 & -\beta\Lambda_{11} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Lambda_{41} & 0 & 0 & \Lambda_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \Lambda_{11}(x - vt) \\ y \\ z \\ \Lambda_{41}x + \Lambda_{44}ct \end{pmatrix}. \tag{2}$$

Galileo and Einstein part ways

So far, we haven't assumed anything about the speed of light, so our results must still include both the Galilean transform and the Lorentz transform.
Let's do the Galilean first:

- Assuming that $t' = t$ gives $\Lambda_{41} = 0$ and $\Lambda_{44} = 1$.
- Assuming that measuring rods have the same length in both frames implies $\Lambda_{11} = 1$.
- This implies the Galilean transformation matrix:

$$\mathbf{G}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Derivation of Lorentz transform (cont'd)

Let's revert to equation (1) and assume that light has same speed c in both frames.

- Imagine a light flash created at $\mathbf{x} = (0, 0, 0, 0)$ expanding with speed c in all directions, creating an expanding spherical wavefront of radii ct and ct' in the two frames. This light cone (a cone in 4D spacetime) is described by

$$x^2 + y^2 + z^2 - (ct)^2 = 0 \tag{3}$$

and

$$x'^2 + y'^2 + z'^2 - (ct')^2 = 0 \tag{4}$$

in the two frames.

- Substiting equation (2) into the last equation gives

$$\Lambda_{11}^2(x-vt)^2+y^2+z^2-(\Lambda_{41}x+\Lambda_{44}ct)^2.$$

- Rearranging terms:

$$(\Lambda_{11}^2-\Lambda_{41}^2)x^2+y^2+z^2-(\Lambda_{44}-\beta^2\Lambda_{11}^2)(ct)^2-2(\beta\Lambda_{11}^2+\Lambda_{41}\Lambda_{44})ctx=0.$$

- So the light cone is where this quadratic polynomial in x, y, z and t vanishes.

- This polynomial will vanish on the same cone as the polynomial of equation (3) only if the two polynomials are identical, *i.e.*, if

$$\begin{cases} \Lambda_{11}^2-\Lambda_{41}^2=1, \\ \Lambda_{44}^2-\beta^2\Lambda_{11}^2=1, \\ \beta\Lambda_{11}^2+\Lambda_{41}\Lambda_{44}=0. \end{cases}$$

- Solve:

$$\begin{cases} \Lambda_{41}^2 = \Lambda_{11}^2 - 1, \\ \Lambda_{44}^2 = \beta^2 \Lambda_{11}^2 + 1, \\ 0 = \beta^2 \Lambda_{11}^4 - \Lambda_{41}^2 \Lambda_{44}^2 = \ldots = 1 - (1 - \beta^2) \Lambda_{11}^2 \end{cases}$$

Solution:

$$\begin{cases} \Lambda_{11} = \gamma, \\ \Lambda_{44} = \gamma, \\ \Lambda_{41} = -\beta\gamma, \\ \Lambda_{14} = -\beta\gamma, \end{cases} ,$$

where we have defined

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

- (We don't care about the 2nd solution with $\Lambda_{11} = -\gamma$, which corresponds to flipping the sign of t and x, "TP".)

- We're done! The Lorentz transformation is

$$\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma(x - \beta ct) \\ y \\ z \\ \gamma(ct - \beta x) \end{pmatrix}.$$

- Compare to Einstein's 1905 paper

Transformation toolbox: summary

- Lorentz transformation:

$$\mathbf{\Lambda}(\hat{\mathbf{x}}v) = \left(\begin{array}{cccc} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{array} \right),$$

i.e.,

$$\left(\begin{array}{c} x' \\ y' \\ z' \\ ct' \end{array} \right) = \left(\begin{array}{c} \gamma(x - \beta ct) \\ y \\ z \\ \gamma(ct - \beta x) \end{array} \right).$$

- This implies all these equations, derived below

- Inverse Lorentz transformation:

$$\boldsymbol{\Lambda}(\mathbf{v})^{-1} = \boldsymbol{\Lambda}(-\mathbf{v})$$

- Addition of parallel velocities:

$$\boldsymbol{\Lambda}(v_1)\boldsymbol{\Lambda}(v_2) = \boldsymbol{\Lambda}\left(\frac{v_1 + v_2}{1 + v_1 v_2}\right)$$

- Addition of arbitrary velocities:

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \\ u_y &= \frac{u'_y \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}} \\ u_z &= \frac{u'_z \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}} \end{aligned}$$

- Boosts as generalized rotations:

$$\mathbf{\Lambda}(-v) = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix},$$

where $\eta \equiv \tanh^{-1}\beta$

- All Lorentz matrices $\mathbf{\Lambda}$ satisfy

$$\mathbf{\Lambda}^t \boldsymbol{\eta} \mathbf{\Lambda} = \boldsymbol{\eta},$$

where the Minkowski metric is

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

- All Lorentz transforms leave the interval

$$\Delta s^2 \equiv \Delta \mathbf{x}^t \boldsymbol{\eta} \Delta x = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2$$

invariant

- Wave 4-vector

$$\mathbf{K} \equiv \gamma_u \begin{pmatrix} k_x \\ k_y \\ k_z \\ w/c \end{pmatrix},$$

- Velocity 4-vector

$$\mathbf{U} \equiv \gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix}, \quad \gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

- Aberration:

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$$

- Doppler effect:

$$\omega' = \omega\gamma(1 - \beta\cos\theta)$$

* Light clock movie

"The Lorentz transformation gives all the relations that I've ever needed to use."

Walter Fox Smith