Class 31: Outline

Hour 1:

Concept Review / Overview

PRS Questions – possible exam questions

Hour 2:

Sample Exam

Yell if you have any questions

Exam 3 Topics

- Faraday's Law
- Self Inductance
  - Energy Stored in Inductor/Magnetic Field
- Circuits
  - LR Circuits
  - Undriven (R)LC Circuits
  - Driven RLC Circuits
- Displacement Current
- Poynting Vector

NO: B Materials, Transformers, Mutual Inductance, EM Waves

General Exam Suggestions

- You should be able to complete every problem
  - If you are confused, ask
  - If it seems too hard, you aren't thinking enough
  - Look for hints in other problems
  - If you are doing math, you're doing too much
- Read directions completely (before & after)
- Write down what you know before starting
- Draw pictures, define (label) variables
  - Make sure that unknowns drop out of solution
- Don't forget units!

Maxwell's Equations

$$\oint\!\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0}$$ (Gauss's Law)

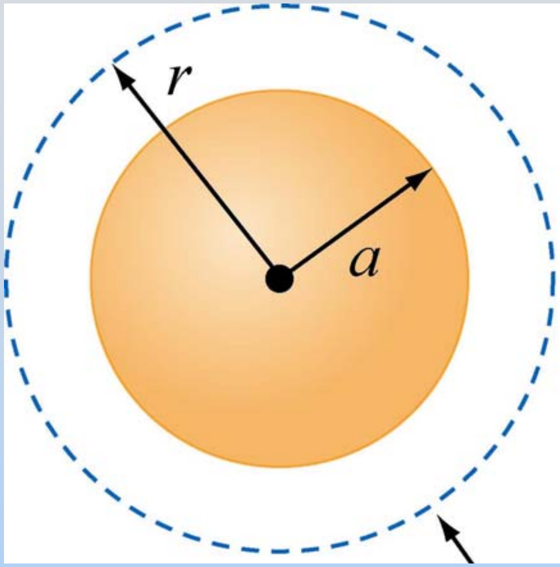
$$\oint_C \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$ (Faraday's Law)

$$\oint\!\!\!\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$ (Magnetic Gauss's Law)

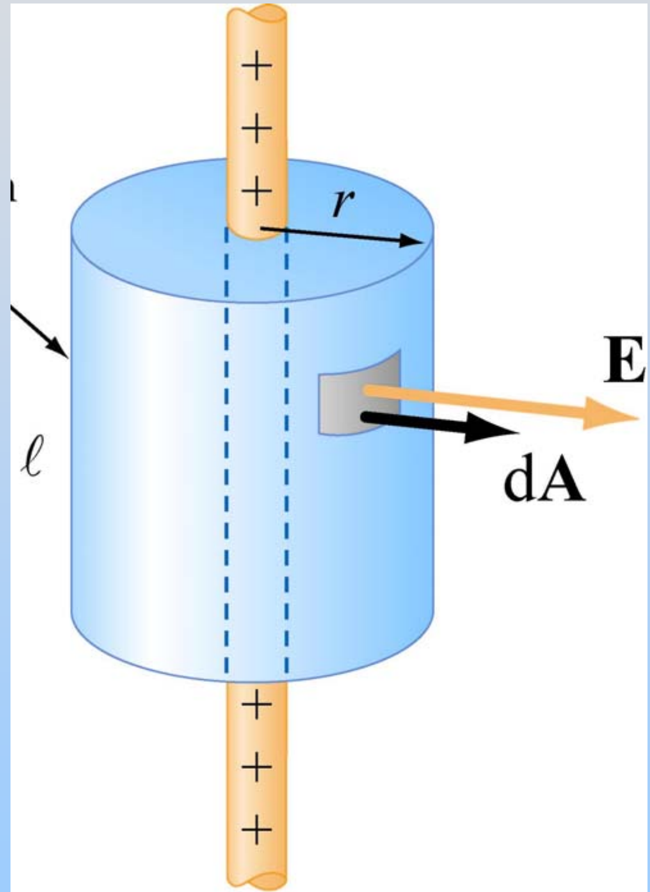
$$\oint_C \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$ (Ampere-Maxwell Law)

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$$ (Lorentz force Law)

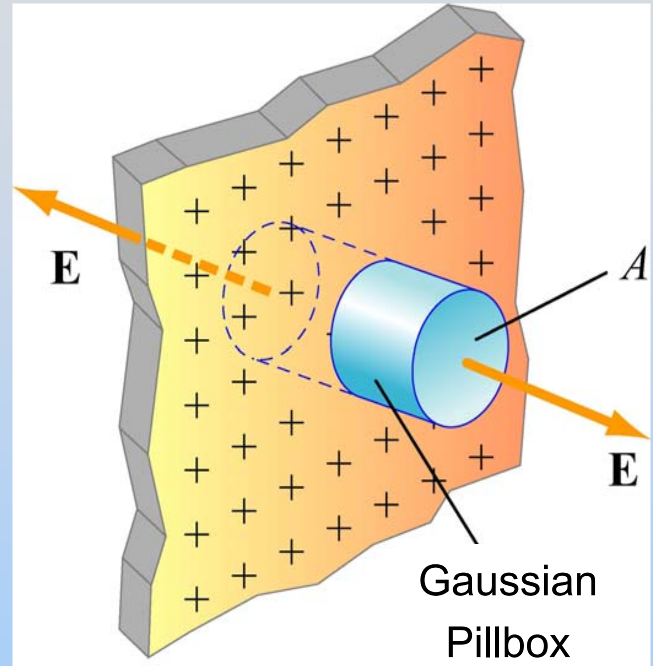
Gauss's Law: $\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$



Spherical Symmetry



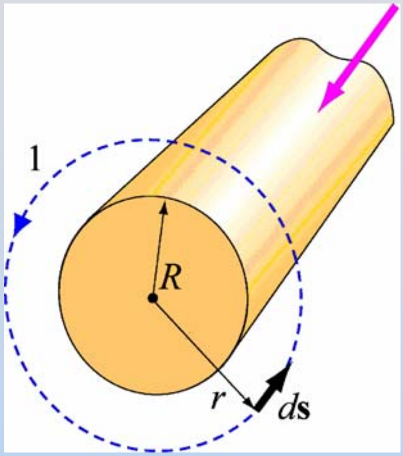
Cylindrical Symmetry

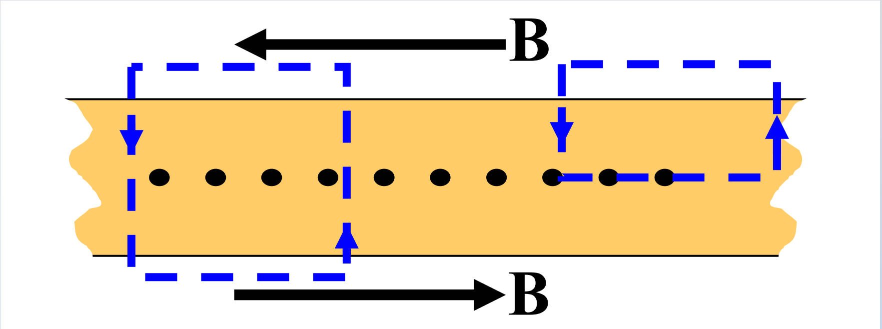


Planar Symmetry

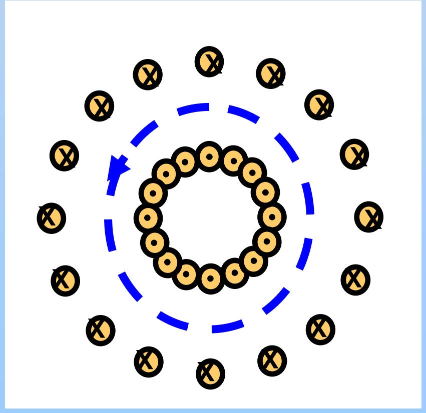
Ampere's Law: $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}$

Long Circular Symmetry



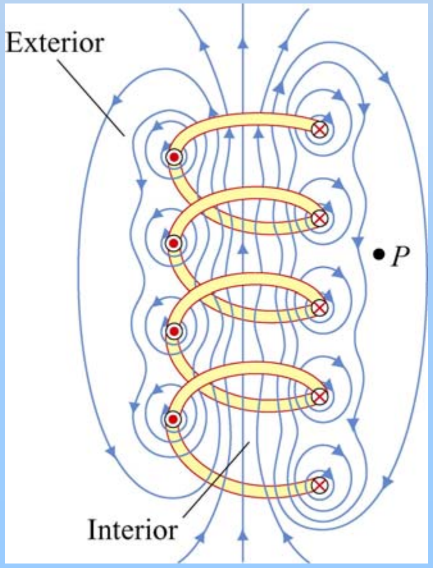


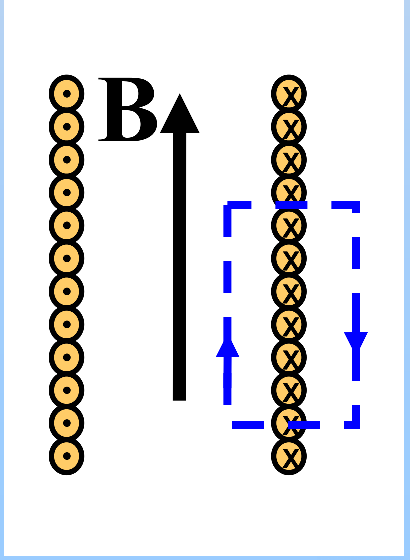
(Infinite) Current Sheet



Torus/Coax

Solenoid = 2 Current Sheets





Faraday's Law of Induction

$$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -N\frac{d\Phi_B}{dt}$$

$$= -N\frac{d}{dt}\left(BA\cos\theta\right)$$

Moving bar, entering field

Ramp B

Rotate area in field

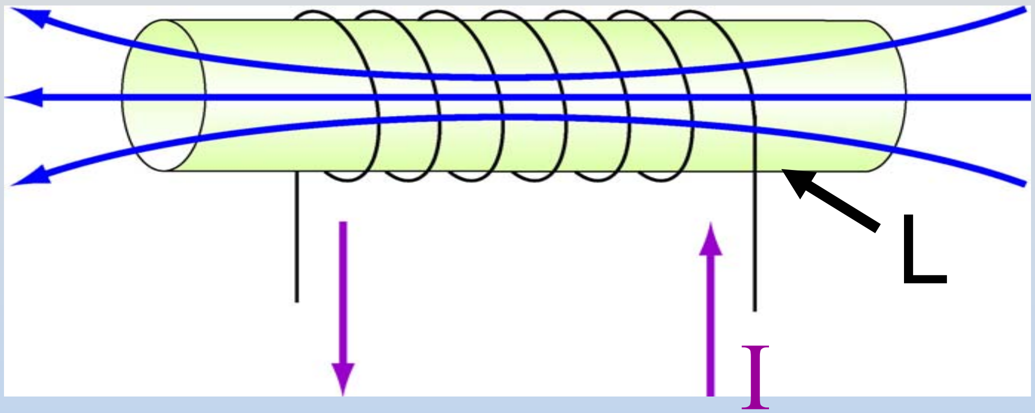
Lenz's Law:

Induced EMF is in direction that ***opposes*** the change in flux that caused it

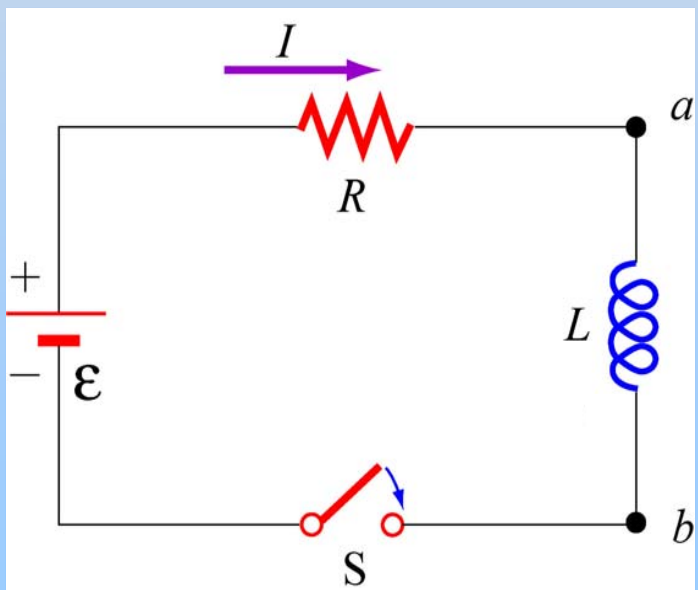
PRS Questions: Faraday's & Lenz's Law

Class 21

Self Inductance & Inductors



$$L = \frac{N\Phi}{I}$$



When traveling in direction of current:

$$\varepsilon = -L\frac{dI}{dt}$$

Notice: This is called "Back EMF"

It is just Faraday's Law!

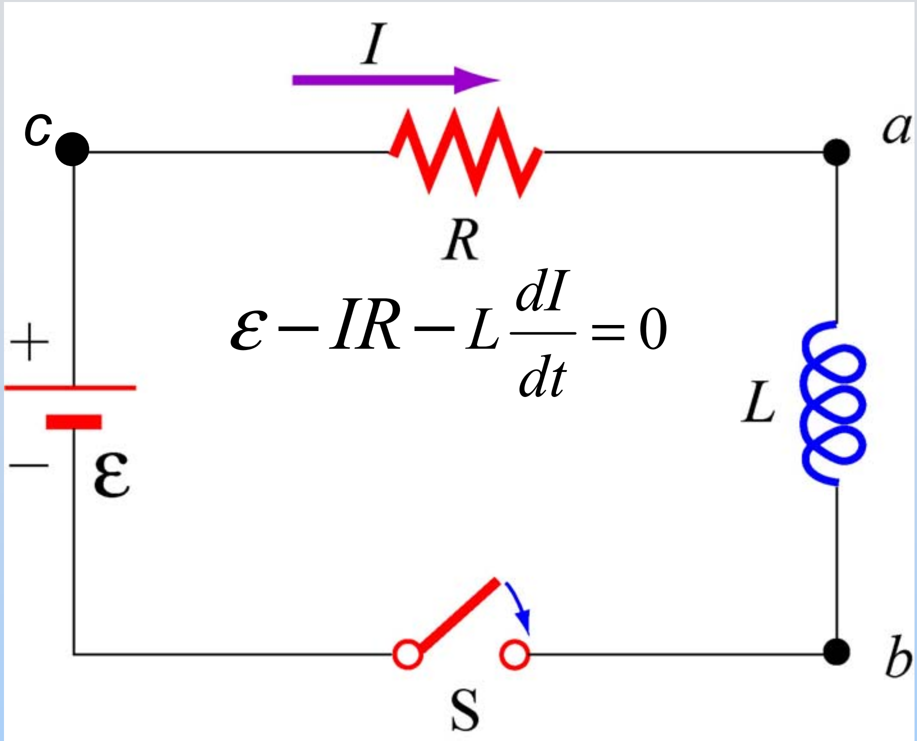
Energy Stored in Inductor

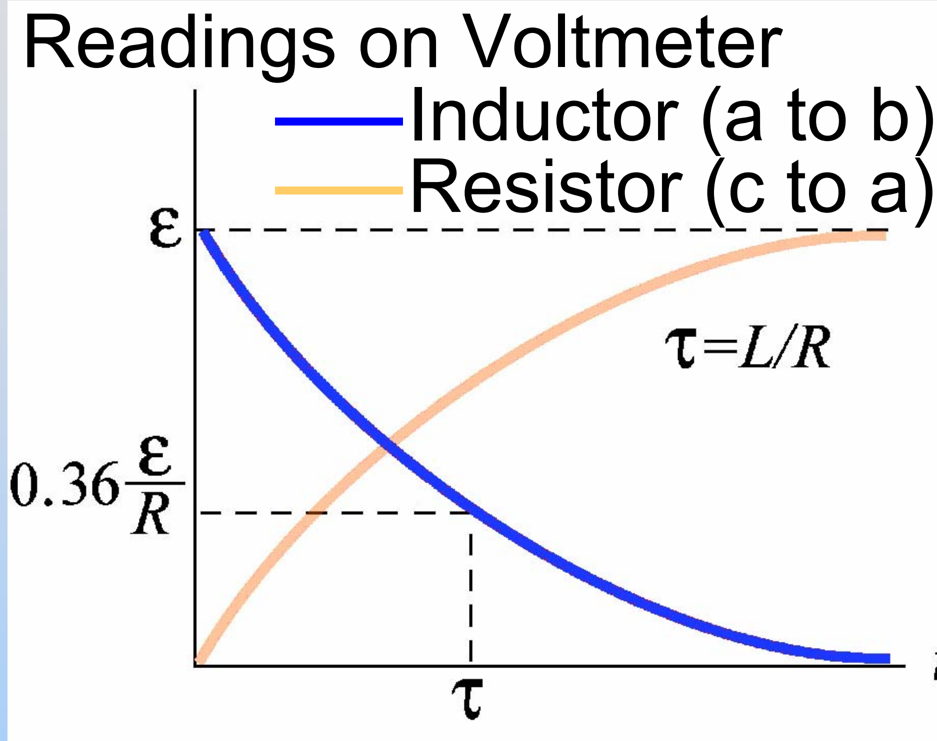
$$U_L = \frac{1}{2} L I^2$$

Energy is stored in the magnetic field:

$$u_B = \frac{B^2}{2\mu_o}$$: Magnetic Energy Density

LR Circuit



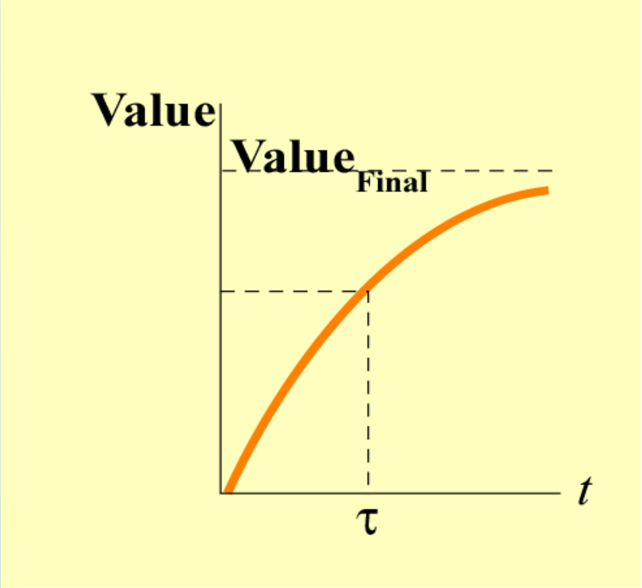


t=0⁺: Current is trying to change. Inductor works as hard as it needs to in order to stop it

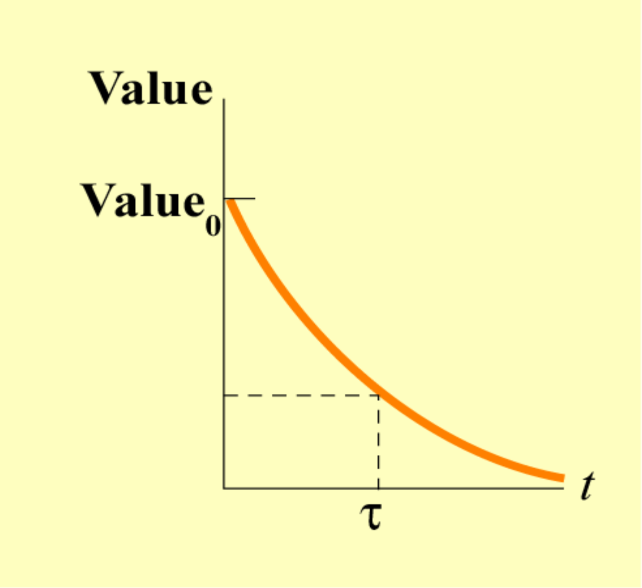
t=∞: Current is steady. Inductor does nothing.

General Comment: LR/RC

All Quantities Either:



$$\text{Value}(t) = \text{Value}_{\text{Final}}\left(1 - e^{-t/\tau}\right)$$



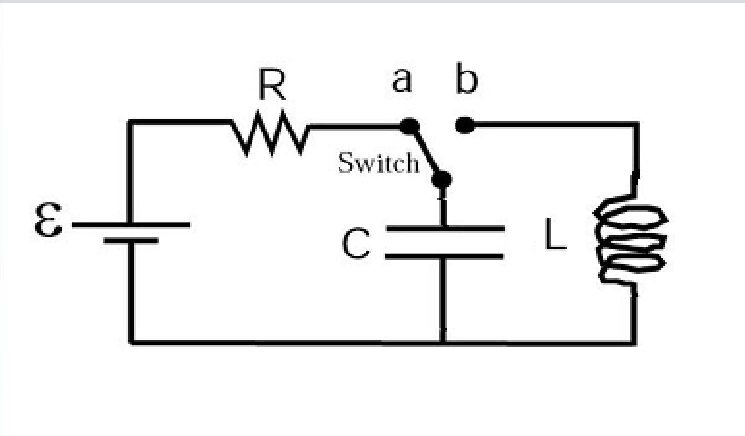
$$\text{Value}(t) = \text{Value}_0 e^{-t/\tau}$$

τ can be obtained from differential equation (prefactor on d/dt) e.g. τ = L/R or τ = RC

PRS Questions: Inductors & LR Circuits

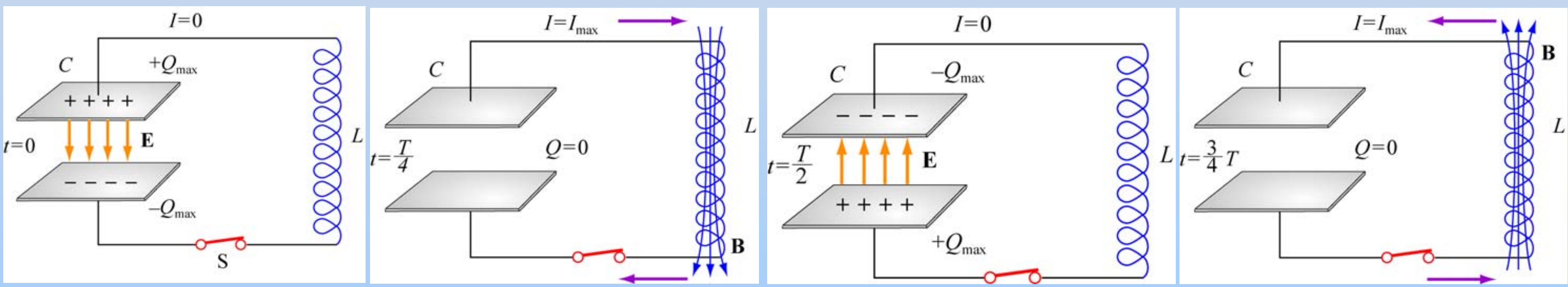
Classes 23, 25

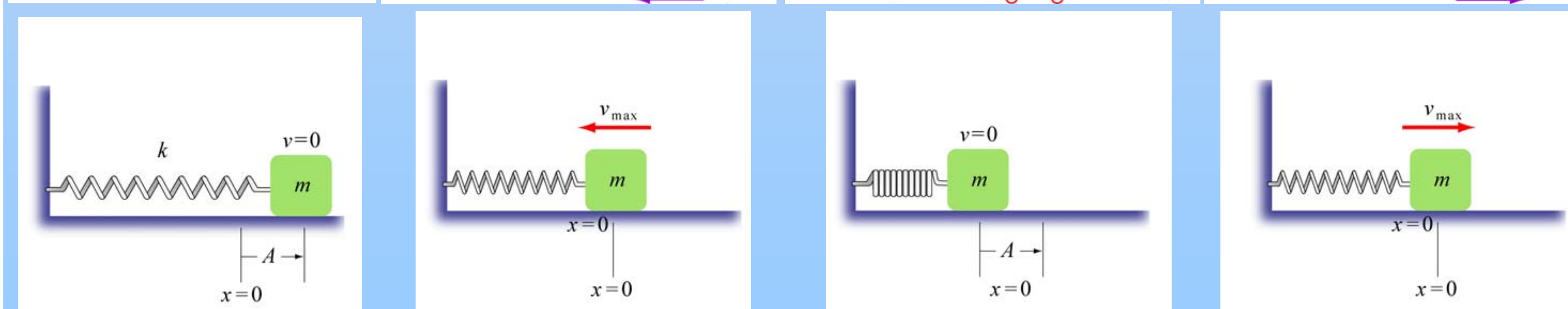
Undriven LC Circuit



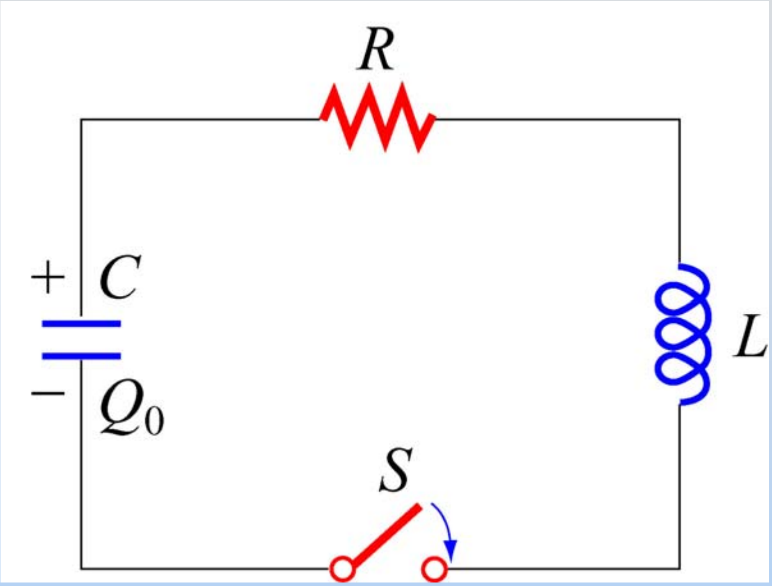
Oscillations: From charge on capacitor (Spring) to current in inductor (Mass)

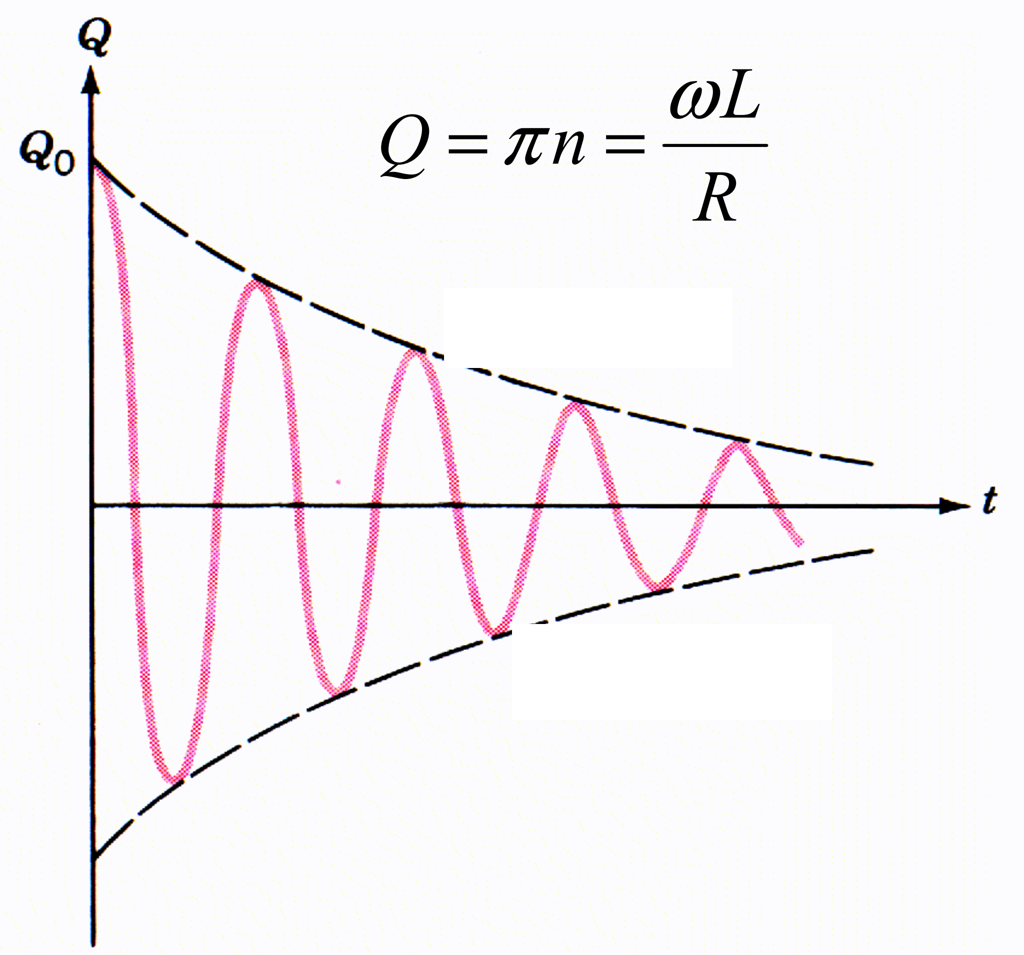
$$\omega_0 = \frac{1}{\sqrt{LC}}$$





Damped LC Oscillations





Resistor dissipates energy and system rings down over time

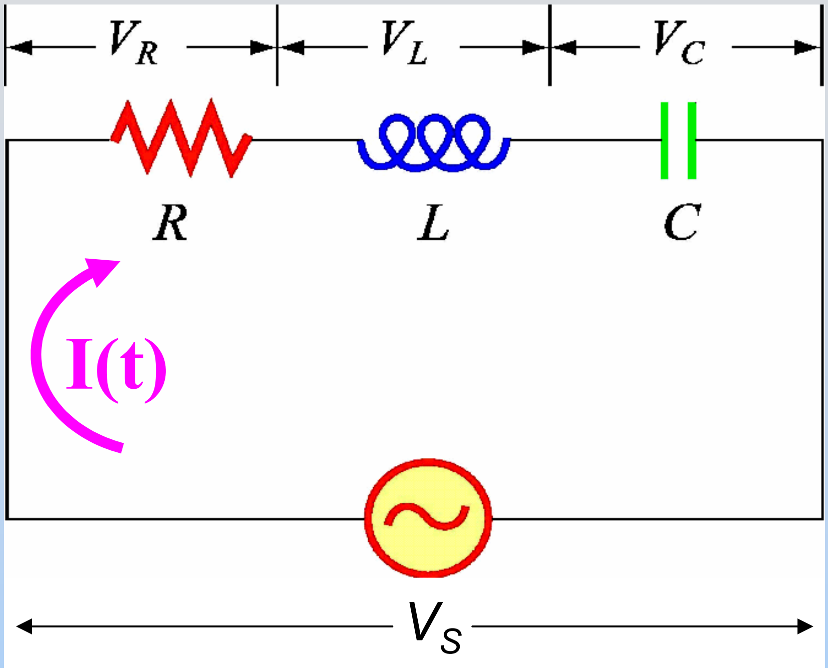
PRS Questions: Undriven RLC Circuits

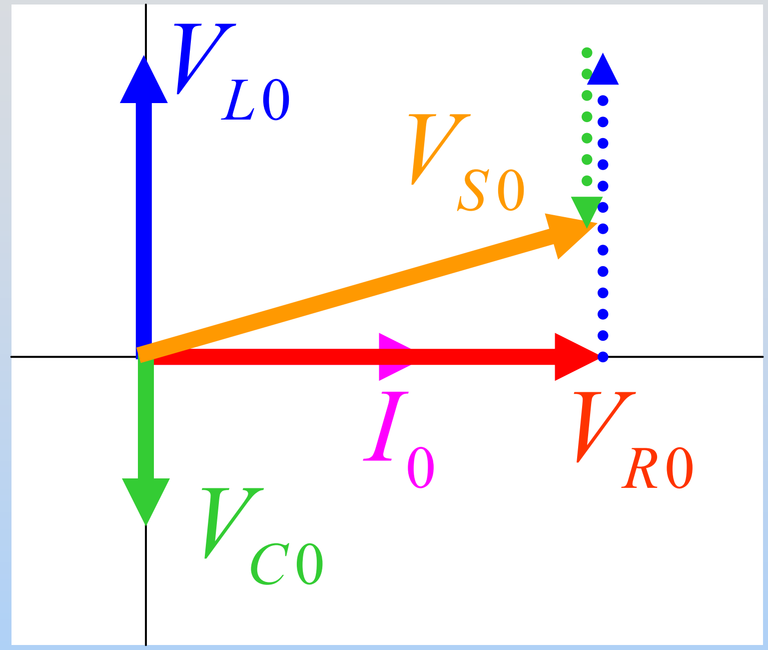
Class 25

AC Circuits: Summary

Element	V vs I_0	Current vs. Voltage	Resistance-Reactance (Impedance)
Resistor	$V_{0R} = I_0 R$	In Phase	$R = R$
Capacitor	$V_{0C} = \frac{I_0}{\omega C}$	Leads (90º)	$X_C = \frac{1}{\omega C}$
Inductor	$V_{0L} = I_0 \omega L$	Lags (90º)	$X_L = \omega L$

Driven RLC Series Circuit

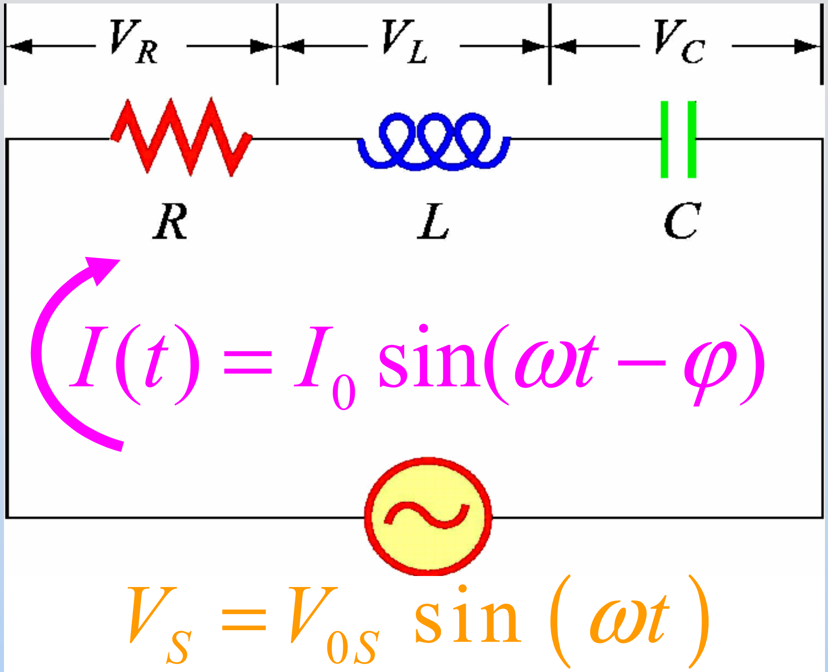


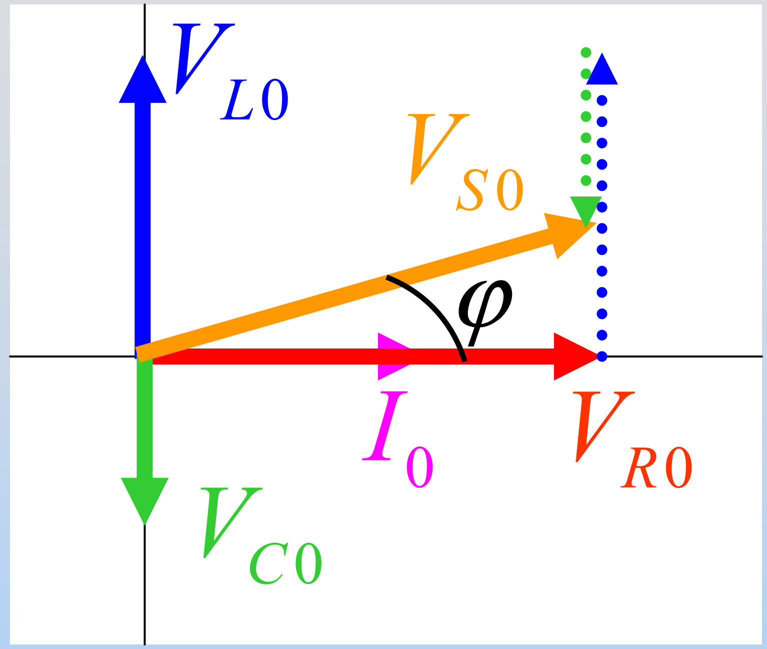


Now Solve: $V_S = V_R + V_L + V_C$

Now we just need to read the phasor diagram!

Driven RLC Series Circuit





$$V_{S0} = \sqrt{V_{R0}{}^2 + (V_{L0} - V_{C0})^2} = I_0\sqrt{R^2 + (X_L - X_C)^2} \equiv I_0 Z$$

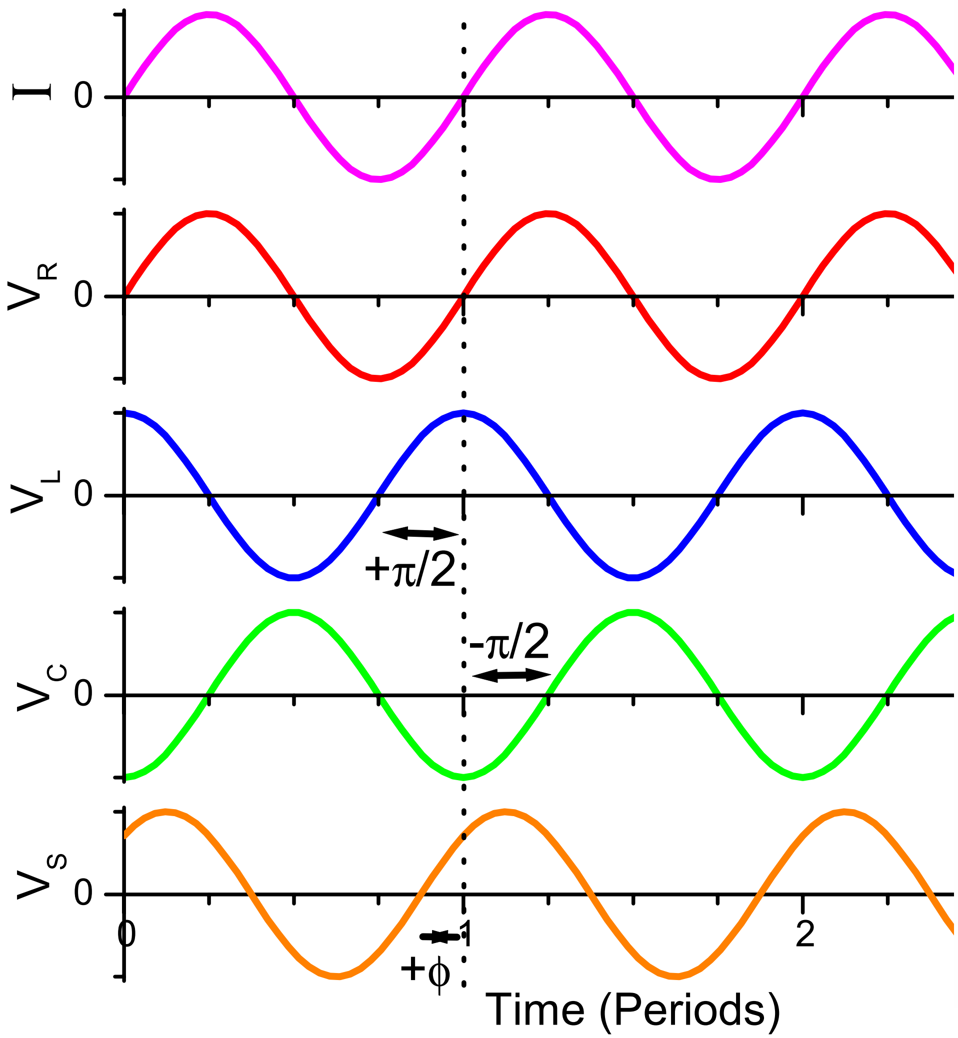
$$I_0 = \frac{V_{S0}}{Z}$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Impedance

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Plot I, V's vs. Time



$$I(t) = I_0 \sin(\omega t)$$

$$V_R(t) = I_0 R \sin(\omega t)$$

$$V_L(t) = I_0 X_L \sin\left(\omega t + \tfrac{\pi}{2}\right)$$

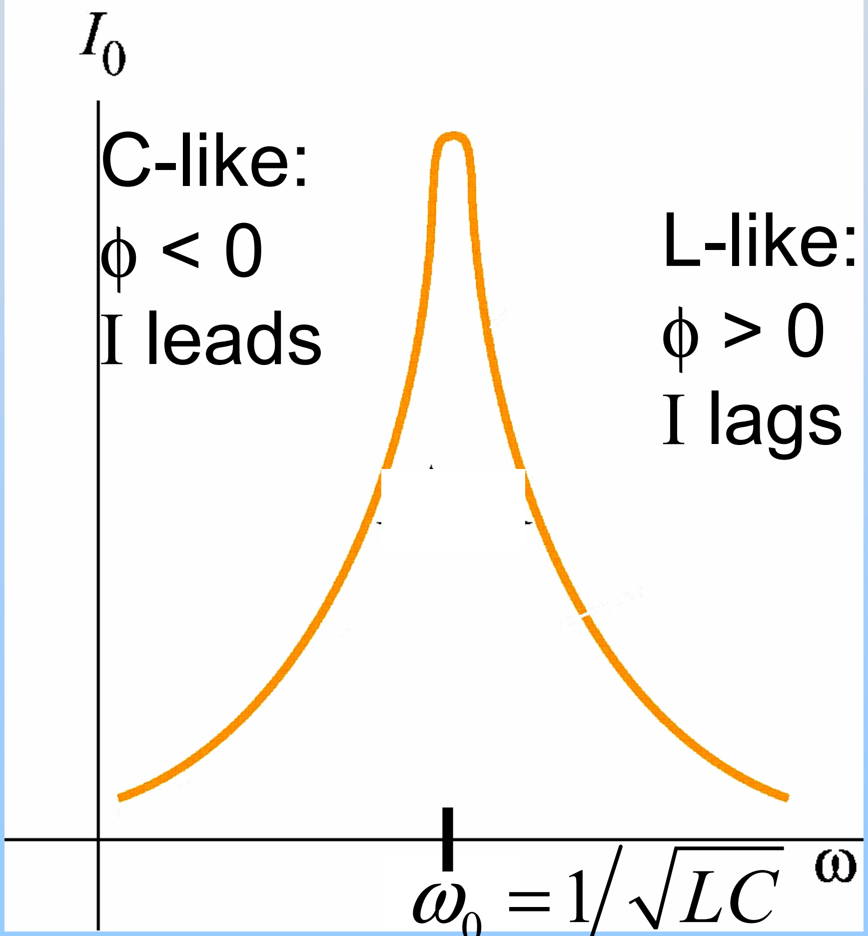
$$V_C(t) = I_0 X_C \sin\left(\omega t - \tfrac{\pi}{2}\right)$$

$$V_S(t) = V_{S0} \sin(\omega t + \varphi)$$

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$



On resonance:

I_0 is max; $X_L = X_C$; Z=R;

$\phi = 0$; Power to R is max

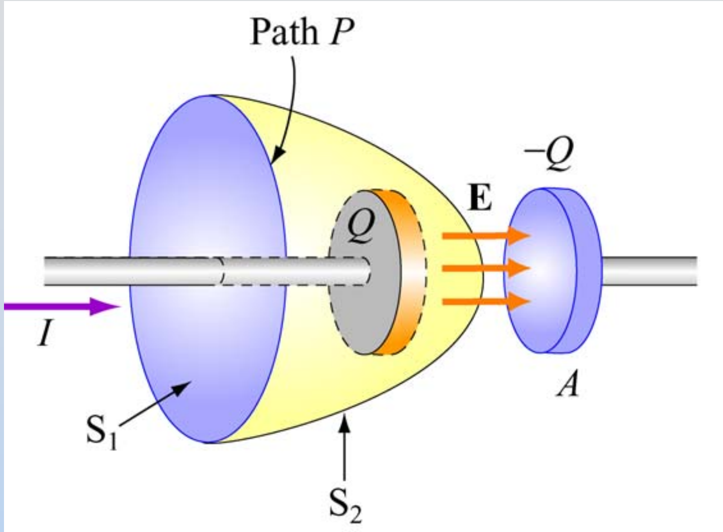
Average Power: Resistor

$$
\begin{aligned}
<P> &= <I^2(t)R> \\
&= <I_0^2 \sin^2(\omega t - \varphi) R> \\
&= I_0^2 R <\sin^2(\omega t - \varphi)> \\
&= I_0{}^2 R\left(\tfrac{1}{2}\right)
\end{aligned}
$$

PRS Questions: Driven RLC Circuits

Class 26

Displacement Current



$$E = \frac{Q}{\varepsilon_0 A} \Rightarrow Q = \varepsilon_0 E A = \varepsilon_0 \Phi_E$$

$$\frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} \equiv I_d$$

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 (I_{encl} + I_d)$$

Capacitors,
EM Waves

$$= \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Energy Flow

Poynting vector: $\vec{\mathbf{S}} = \dfrac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0}$

- (Dis)charging C, L
- Resistor (always in)
- EM Radiation

PRS Questions: Displacement/Poynting

Class 28

SAMPLE EXAM:

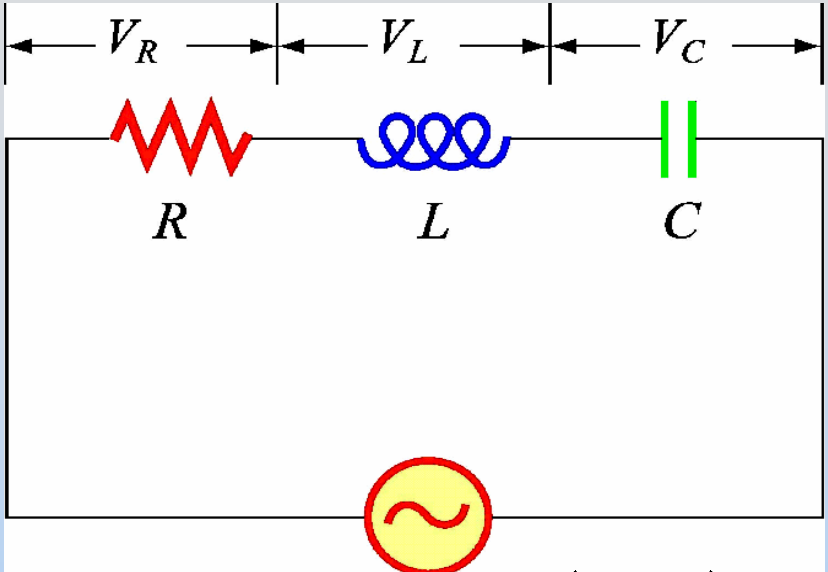
The real exam has 8 concept, 3 analytical questions

Problem 1: RLC Circuit

Consider a circuit consisting of an AC voltage source: $V(t)=V_0\sin(\omega t)$ connected in series to a capacitor C and a coil, which has resistance R and inductance L_0.

1. Write a differential equation for the current in this circuit.
2. What angular frequency ω_{res} would produce a maximum current?
3. What is the voltage across the capacitor when the circuit is driven at this frequency?

Solution 1: RLC Circuit



$$V_S = V_0 \sin(\omega t)$$

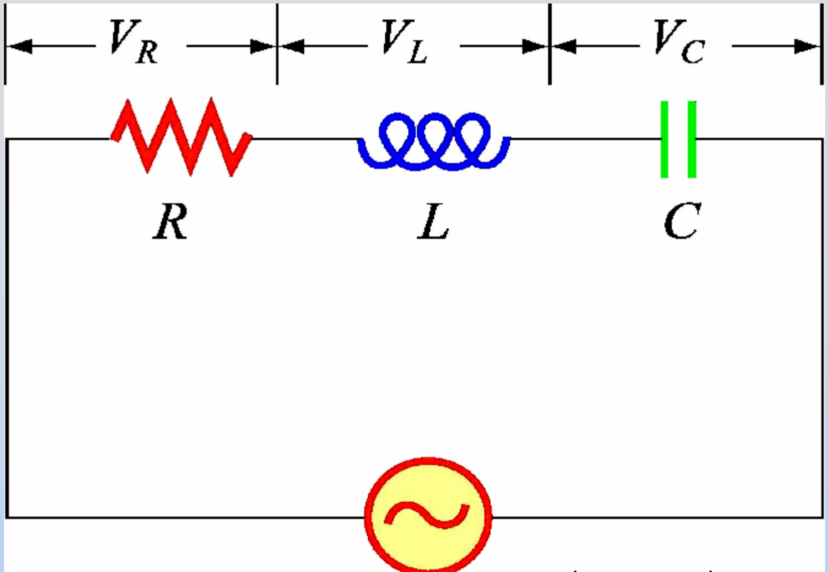
1. Differential Eqn:

$$V_S - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0$$

$$\frac{dI}{dt}R + L\frac{d^2I}{dt^2} + \frac{I}{C} = \frac{d}{dt}V_S = \omega V_0 \cos(\omega t)$$

2. Maximum current on resonance: $\omega_{res} = 1\big/\sqrt{L_0 C}$

Solution 1: RLC Circuit



$$V_S = V_0 \sin(\omega t)$$

3. Voltage on Capacitor

$V_{C0} = I_0 X_C$ What is I_0, X_C?

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{R} \text{ (resonance)}$$

$$X_C = \frac{1}{\omega C} = \frac{\sqrt{L_0 C}}{C} = \sqrt{\frac{L_0}{C}}$$

$$V_{C0} = I_0 X_C = \frac{V_0}{R}\sqrt{\frac{L_0}{C}}$$

$$V = V_{C0}\left(-\cos(\omega t)\right) = V_{C0}\sin\left(\omega t - \tfrac{\pi}{2}\right)$$

Problem 1, Part 2: RLC Circuit

Continue considering that LRC circuit.

Insert an iron bar into the coil. Its inductance changes by a factor of 5 to $L=L_{core}$

4. Did the inductance increase or decrease?
5. Is the new resonance frequency larger, smaller or the same as before?
6. Now drive the new circuit with the original ω_{res}. Does the current peak before, after, or at the same time as the supply voltage?

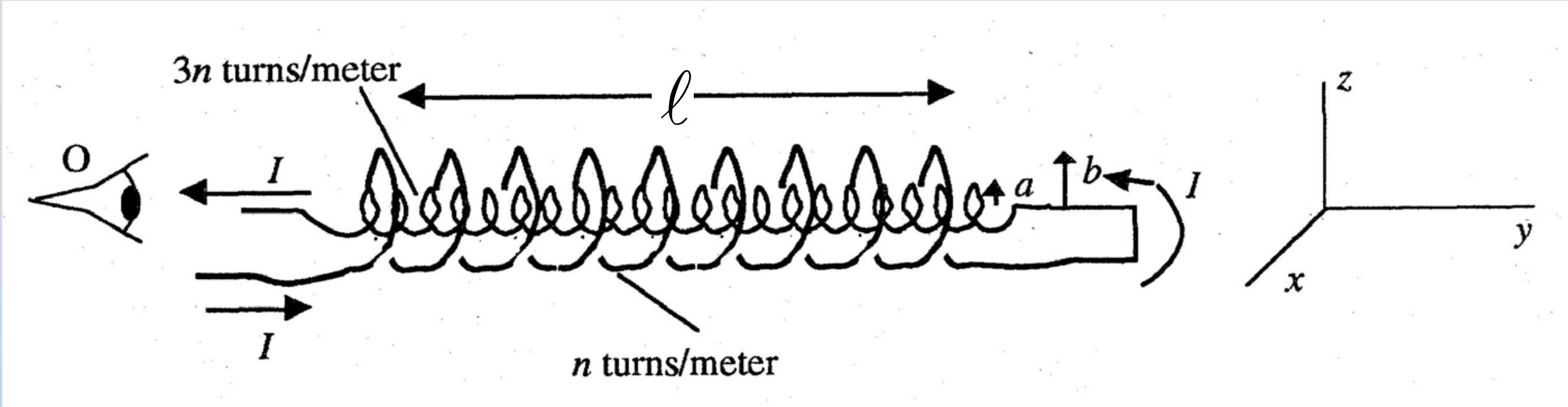
Solution 1, Part 2: RLC Circuit

4. Putting in an iron core INCREASES the inductance

5. The new resonance frequency is smaller

6. If we drive at the original resonance frequency then we are now driving ABOVE the resonance frequency. That means we are inductor like, which means that the current lags the voltage.

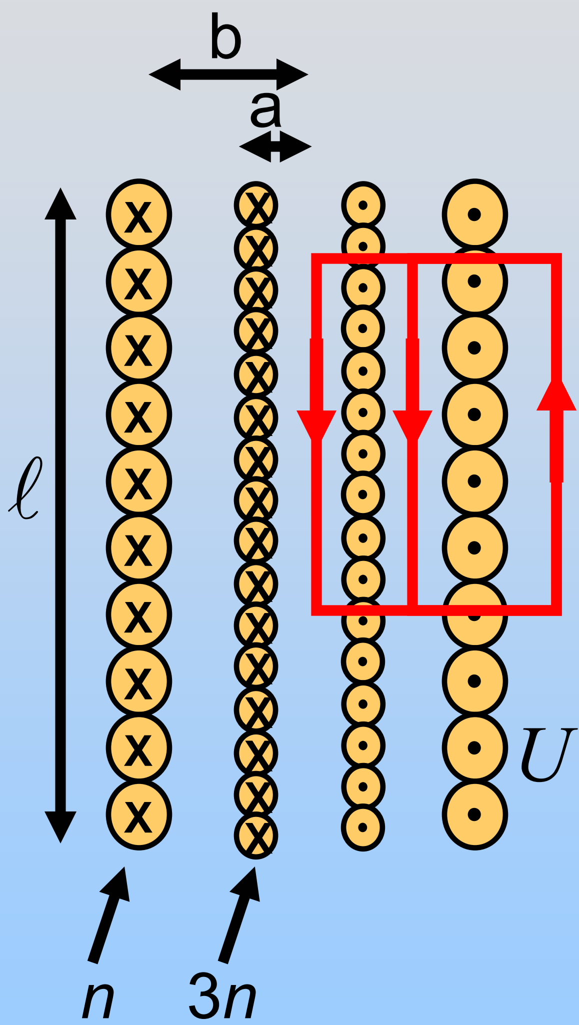
Problem 2: Self-Inductance



The above inductor consists of two solenoids (radius b, n turns/meter, and radius a, 3n turns/meter) attached together such that the current pictured goes counter-clockwise in both of them according to the observer.

What is the self inductance of the above inductor?

Solution 2: Self-Inductance



Inside Inner Solenoid:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl = \mu_0 \left(nlI + 3nlI \right)$$

$$\Rightarrow B = 4\mu_0 nI$$

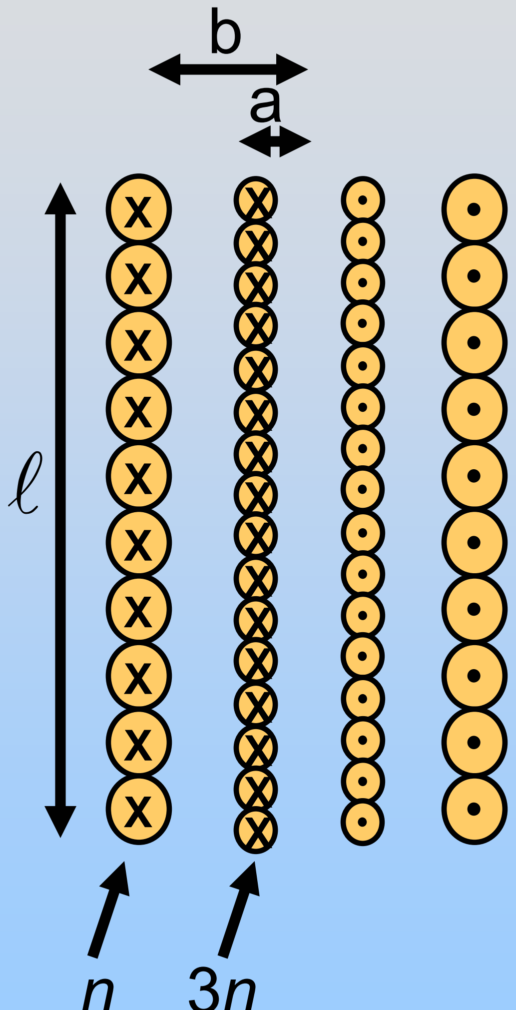
Between Solenoids:

$$B = \mu_0 nI$$

$$U = \frac{B^2}{2\mu_o} \bullet \text{Volume}$$

$$= \frac{\left(4\mu_0 nI\right)^2}{2\mu_o} \pi a^2 \ell + \frac{\left(\mu_0 nI\right)^2}{2\mu_o} \pi \left(b^2 - a^2\right) \ell$$

Solution 2: Self-Inductance



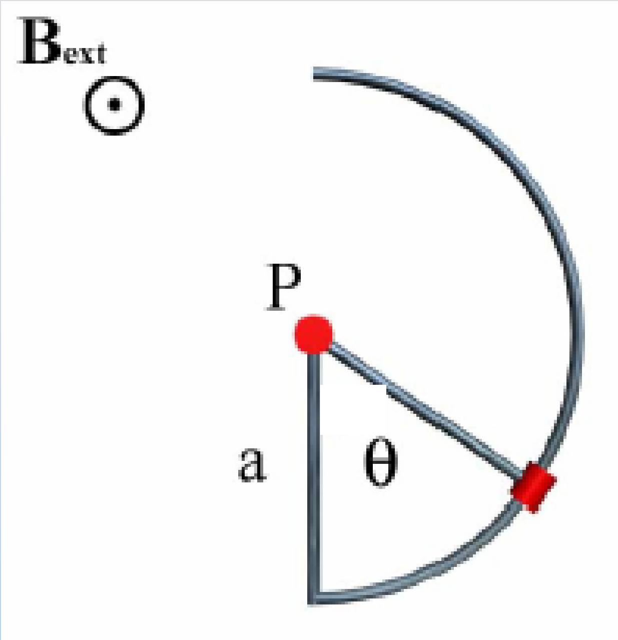
$$U = \frac{(\mu_0 n I)^2}{2\mu_o} \pi \ell \left\{ 15a^2 + b^2 \right\}$$

$$U = \tfrac{1}{2} L I^2$$

$$\Rightarrow L = \frac{(\mu_0 n)^2}{\mu_o} \pi \ell \left\{ 15a^2 + b^2 \right\}$$

Could also have used: $L = \dfrac{N\Phi}{I}$

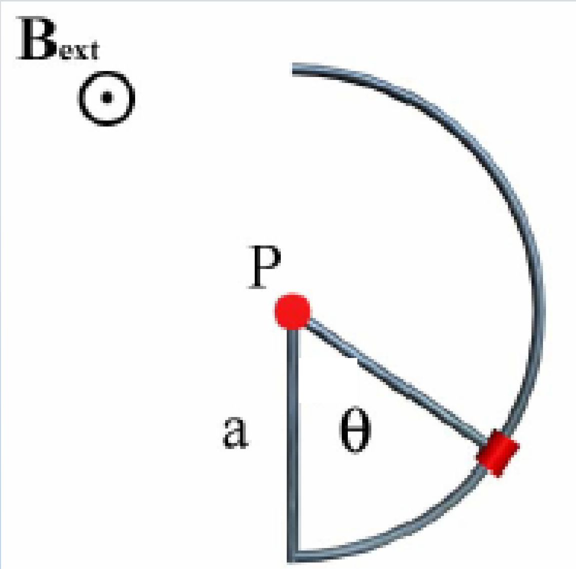
Problem 3: Pie Wedge



Consider the following pie shaped circuit. The arm is free to pivot about the center, P, and has mass *m* and resistance *R*.

1. If the angle θ decreases in time (the bar is falling), what is the direction of current?
2. If $\theta = \theta(t)$, what is the rate of change of magnetic flux through the pie-shaped circuit?

Solution 3: Pie Wedge



1) Direction of I?

Lenz's Law says: try to oppose decreasing flux

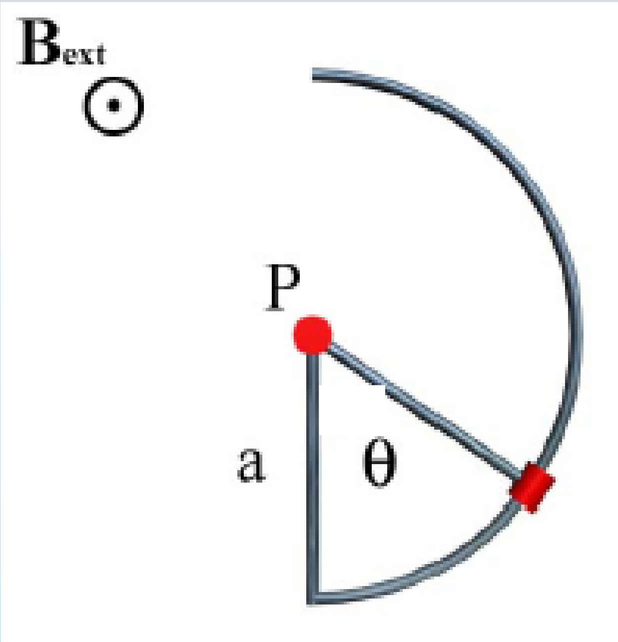
I Counter-Clockwise (B out)

2) θ = θ(t), rate of change of magnetic flux?

$$A = \pi a^2 \left(\frac{\theta}{2\pi} \right) = \frac{\theta a^2}{2} \qquad \frac{d\Phi_B}{dt} = \frac{d}{dt}(BA) = B\frac{d}{dt}\frac{\theta a^2}{2}$$

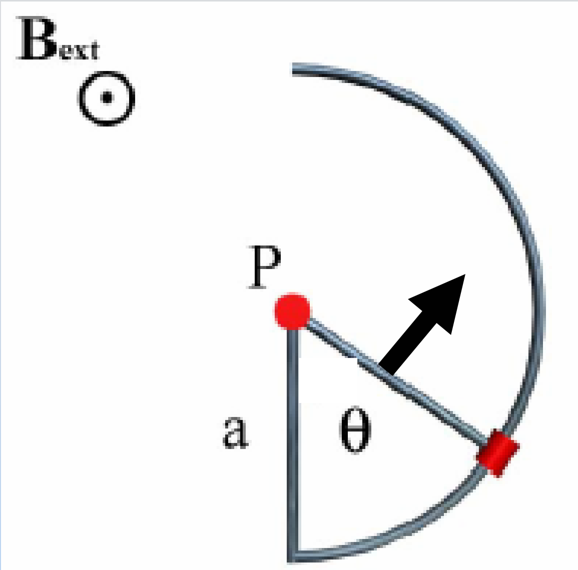
$$= \frac{Ba^2}{2}\frac{d\theta}{dt}$$

Problem 3, Part 2: Pie Wedge



3. What is the magnetic force on the bar (magnitude and direction – indicated on figure)
4. What torque does this create about P? (HINT: Assume force acts at bar center)

Solution 3, Part 2: Pie Wedge



3) Magnetic Force?

$$d\vec{\mathbf{F}} = Id\vec{\mathbf{s}} \times \vec{\mathbf{B}} \qquad F = IaB$$

$$I = \frac{\varepsilon}{R} = \frac{1}{R}\frac{d\Phi_B}{dt} = \frac{1}{R}\frac{Ba^2}{2}\frac{d\theta}{dt}$$

$$\boxed{F = \frac{B^2a^3}{2R}\frac{d\theta}{dt}}$$

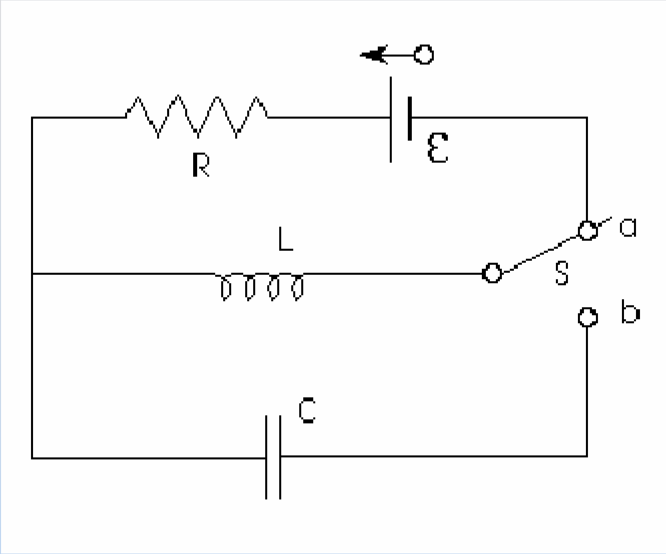
(Dir. as pictured)

4) Torque?

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \Rightarrow \tau = \frac{a}{2}F \boxed{= \frac{B^2a^4}{4R}\frac{d\theta}{dt}}$$

(out of page)

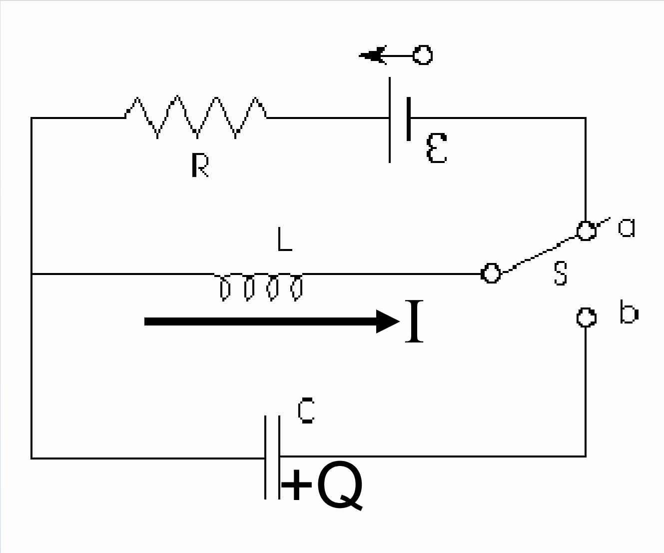
Problem 4: RLC Circuit



The switch has been in position ***a*** for a long time. The capacitor is uncharged.

1. What energy is currently stored in the magnetic field of the inductor?
2. At time $t = 0$, the switch S is thrown to position *b.* By applying Faraday's Law to the bottom loop of the above circuit, obtain a differential equation for the behavior of charge Q on the capacitor with time.

Solution 4: RLC Circuit



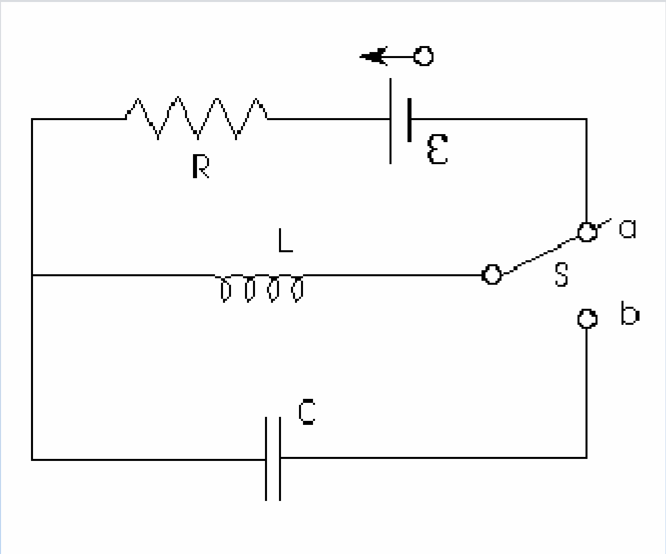
1. Energy Stored in Inductor

$$U = \frac{1}{2} L I^2 = \frac{1}{2} L \left(\frac{\varepsilon}{R} \right)^2$$

2. Write Differential Equation

$$-L \frac{dI}{dt} - \frac{Q}{C} = 0 \qquad I = \frac{dQ}{dt} \Rightarrow L \frac{d^2 Q}{dt^2} + \frac{Q}{C} = 0$$

Problem 4, Part 2: RLC Circuit



3. Write down an explicit solution for *Q(t)* that satisfies your differential equation above and the initial conditions of this problem.

4. How long after $t = 0$ does it take for the electrical energy stored in the capacitor to reach its first maximum, in terms of the quantities given? At that time, what is the energy stored in the inductor? In the capacitor?

Solution 4: RLC Circuit

3. Solution for $Q(t)$: $Q(t) = Q_{\max} \sin(\omega t)$

$$\boxed{\omega = \frac{1}{\sqrt{LC}}} \qquad \omega Q_{\max} = I_0 = \frac{\varepsilon}{R} \Rightarrow \boxed{Q_{\max} = \frac{\varepsilon \sqrt{LC}}{R}}$$

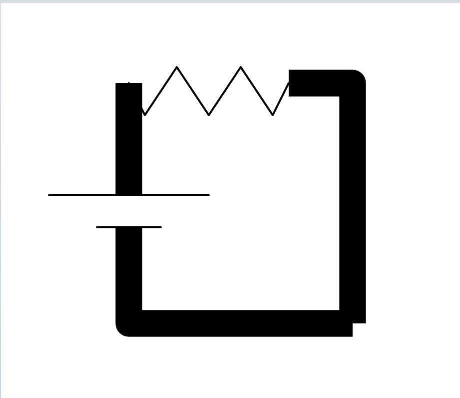
4. Time to charge capacitor

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} \Rightarrow \boxed{T_{\text{Charge}} = \frac{T}{4} = \frac{\pi\sqrt{LC}}{2}}$$

Energy in inductor = 0

Energy in capacitor = Initial Energy: $U = \frac{1}{2} L \left(\frac{\varepsilon}{R}\right)^2$

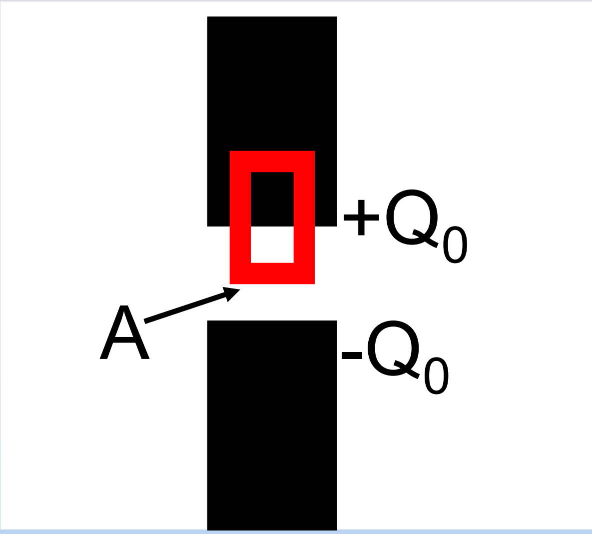
Problem 5: Cut Circuit



Consider the circuit at left: A battery (EMF ε) and a resistor wired with very thick wire of radius *a*.
At time t=0, a thin break is made in the wire (thickness *d*).

1. After a time $t = t_0$, a charge $Q = Q_0$ accumulates at the top of the break and $Q = -Q_0$ at the bottom. What is the electric field inside the break?
2. What is the magnetic field, B, inside the break as a function of radius $r<a$?

Solution 5: Cut Circuit



1. Cut looks like capacitor.

Use Gauss to find electric field:

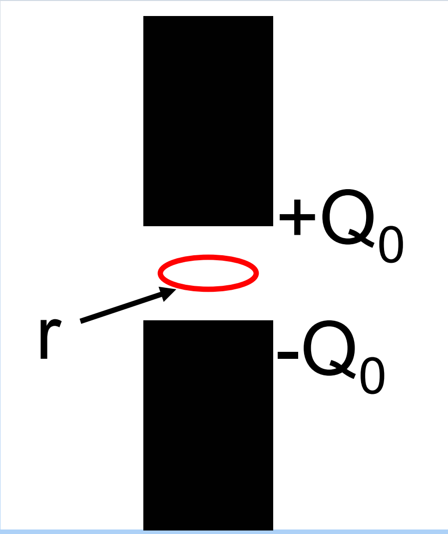
$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{E = \frac{\sigma}{\varepsilon_0} = \frac{Q_0}{\pi a^2 \varepsilon_0} \text{ down}}$$

Solution 5: Cut Circuit

$$E = \frac{Q_0}{\pi a^2 \varepsilon_0}$$

2. Find B field using Ampere's Law



$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} E\pi r^2 = \varepsilon_0 \pi r^2 \frac{dE}{dt}$$

$$= \varepsilon_0 \pi r^2 \frac{d}{dt}\left(\frac{Q_0}{\pi a^2 \varepsilon_0}\right) = \frac{r^2}{a^2}\frac{dQ_0}{dt}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B 2\pi r = \mu_0 \left(I_{enc} + I_d\right) = \mu_0 I_d = \mu_0 \frac{r^2}{a^2}\frac{dQ_0}{dt}$$

$$\boxed{B = \frac{\mu_0}{2\pi}\frac{r}{a^2}\frac{dQ_0}{dt} \text{ clockwise}}$$