- ## Geometry and trigonometry

Circle of radius r: circumference = $2\pi r$; area = πr^2.
Sphere of radius r: area = $4\pi r^2$; volume = $4/3\pi r^3$.
Right circular cylinder of radius r and height h:
Area = $2\pi r^2 + 2\pi rh$; volume = $\pi r^2 h$.
Triangle of base b and altitude h: area = 1/2 bh.

Quadratic Formula

If $ax^2 + bx + c = 0,$ then $x = \dfrac{-b \times \sqrt{b^2 - 4ac}}{2a}$

Trigonometric Functions of Angle θ

$sin\theta = y/r \quad cos\theta = x/r$
$tan\theta = y/x \quad cot\theta = x/y$
$sec\theta = r/x \quad csc\theta = r/y$

Trigonometric Identities

$$\sin(\frac{\pi}{2} - \theta) = \cos\theta$$
$$\cos(\frac{\pi}{2} - \theta) = \sin\theta$$
$$\sin\theta / \cos\theta = \tan\theta$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\sec^2\theta - \tan^2\theta = 1$$
$$\csc^2\theta - \cot^2\theta = 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$
$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \pm \beta)$$

- **Expansions**

Binomial:

$$(1 \pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \pm \cdots (x^2 < 1)$$

$$(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)x^2}{2!} \mp \cdots (x^2 < 1)$$

Exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

$$\ln(1 \pm x) = x - \frac{x^2}{2!} + \frac{1}{3!} x^3 - \cdots$$

Trigonometric:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3!} + \frac{2\theta^5}{15} + \cdots$$

- **Derivative and Integrals**

$\frac{dx}{dx}=1$	$\int dx = x$
$\frac{d}{dx}(au) = a\frac{du}{dx}$	$\int (au)dx = a\int udx$
$\frac{d}{dx}(u+v) = \frac{du}{dx}+\frac{dv}{dx}$	$\int (u+v)dx = \int udx + \int vdx$
$\frac{d}{dx}x^m = mx^{m-1}$	$\int x^m dx = \frac{mx^{m-1}}{m+1}(m \neq 1)$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x} = \ln\lvert x\rvert$
$\frac{d}{dx}(uv) = v\frac{du}{dx}+u\frac{dv}{dx}$	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x$
$\frac{d}{dx}\sin x = \cos x$	$\int \sin xdx = -\cos x$
$\frac{d}{dx}\cos x = -\sin x$	$\int \cos xdx = \sin x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \tan xdx = -\ln\cos x$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \sin^2 xdx = \frac{1}{2}x - \frac{1}{4}\sin 2x$
$\frac{d}{dx}\sec x = \tan x\sec x$	$\int \cos^2 xdx = \frac{1}{2}x + \frac{1}{4}\sin 2x$
$\frac{d}{dx}\csc x = -\cot x\csc x$	$\int e^{-ax}dx = -\frac{1}{a}e^{-ax}$
$\frac{d}{dx}e^u = e^u\frac{du}{dx}$	$\int xe^{-ax}dx = -\frac{1}{a^2}(ax+1)e^{-ax}$
$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$	$\int x^2e^{-ax}dx = -\frac{1}{a^3}(a^2x^2+2ax+2)e^{-ax}$
$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$	$\int x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$
	$\int_0^\infty x^{2n}e^{-ax} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$
	$\int \frac{dx}{\sqrt{(x^2\pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2\pm a^2}}$

- **Vector Derivatives**

Cartesian.

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial t}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial t}{\partial z}\hat{\boldsymbol{z}}$

Divergence: $\nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \vec{\boldsymbol{v}} = (\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})\hat{\boldsymbol{x}} + (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x})\hat{\boldsymbol{y}} + (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\hat{\boldsymbol{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical.

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\varphi}}$

Divergence: $\nabla \cdot \vec{\mathbf{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl:

$$\nabla \times \vec{\boldsymbol{v}} = \frac{1}{r\sin\theta}[\frac{\partial(\sin\theta v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}]\hat{\boldsymbol{r}} + \frac{1}{r}[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}]\hat{\boldsymbol{\theta}} + \frac{1}{r}[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}]\hat{\boldsymbol{\varphi}}$$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial t}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial t}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical.

Gradient: $\nabla t = \frac{\partial t}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\varphi}} + \frac{\partial t}{\partial z}\hat{\boldsymbol{z}}$

Divergence: $\nabla \cdot \vec{\mathbf{v}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \vec{\boldsymbol{v}} = [\frac{1}{\rho}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}]\hat{\boldsymbol{\rho}} + [\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial s}]\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}[\frac{\partial(\rho v_\phi)}{\partial\rho} - \frac{\partial v_\rho}{\partial\theta}]\hat{\boldsymbol{z}}$

Laplacian: $\nabla^2 t = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial t}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

- **Spherical and cylindrical coordinates**

Spherical

$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$	$\begin{cases} \hat{\boldsymbol{x}} = \sin\theta\cos\phi\hat{\boldsymbol{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{y}} = \sin\theta\sin\phi\hat{\boldsymbol{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{z}} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}} \end{cases}$
$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases}$	$\begin{cases} \hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\boldsymbol{x}} + \cos\theta\sin\phi\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\varphi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{z}} \end{cases}$

Cylindrical

$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$	$\begin{cases} \hat{\boldsymbol{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\varphi}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$
$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$	$\begin{cases} \hat{\boldsymbol{\rho}} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\varphi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$

- **Vector Identities**

Triple Products

1. $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$
2. $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{C} \cdot \boldsymbol{A}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$

Product Rules

1. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
2. $\nabla(\boldsymbol{A} \cdot \boldsymbol{C}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$
3. $\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A}(\nabla \cdot f)$
4. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$
5. $\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla f)$
6. $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})$

Second Derivatives

1. $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$
2. $\nabla \times (\nabla f) = 0$
3. $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

- **Fundamental Theorems:**

Gradient Theorem: $\int_{\boldsymbol{a}}^{\boldsymbol{b}} (\nabla f) \cdot d\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})$

Divergence Theorem: $\int (\nabla \cdot \boldsymbol{A}) d\tau = \oint \boldsymbol{A} \cdot d\boldsymbol{a}$

Curl Theorem: $\int (\nabla \times \boldsymbol{A}) d\boldsymbol{a} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$