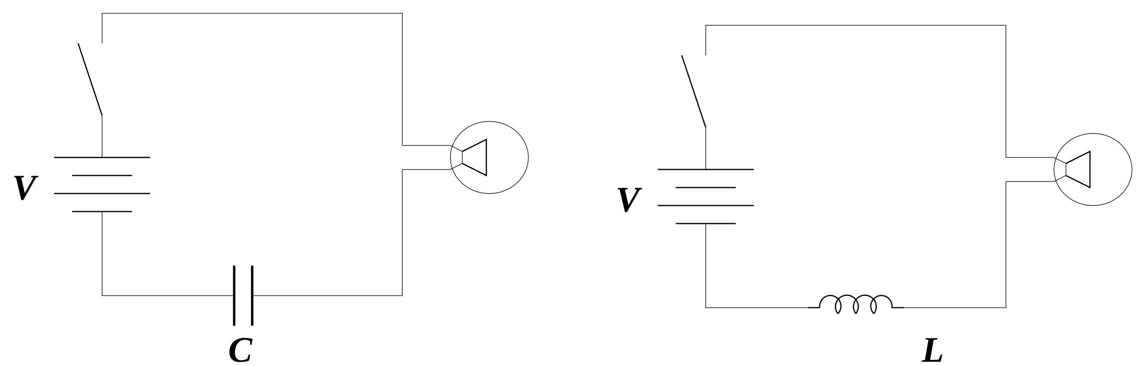
Massachusetts Institute of Technology
Department of Physics
8.022 Spring 2004

Solution to Quiz #2

PROBLEM 1 (9 pts)

Answer the following questions, justifying your answers with one short sentence.

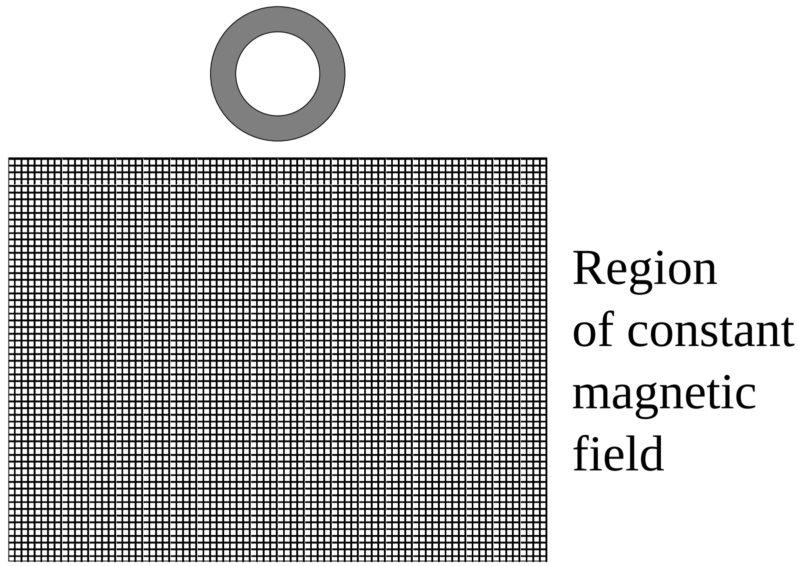
(a) Assume that lightbulbs act as resistors and examine the following two circuits:



Both switches are closed at $t = 0$. Qualitatively describe the behavior of the lightbulbs as time passes.

In the left circuit, current flows until the charge builds on the capacitor, opposing the battery. In the right circuit, the inductor initially opposes current flow — Lenz's law! — and only gradually allows the current to flow. Hence, the left bulb will glow brightly and then fade away. The right bulb will start out dark and gradually get brighter.

(b) A metal ring is dropped (feels uniform gravitational acceleration down) into a region of constant magnetic field, pointing out of the page:

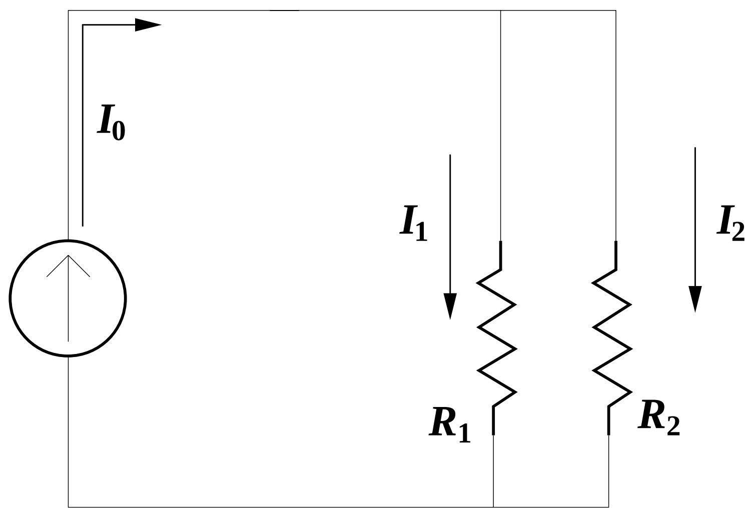


Describe the motion of the ring as it falls. Be sure to contrast the behavior when it is out of the field, when it is partially in the field, and when it is completely in the field.
Lenz's law tells us that an induced current will flow when there is a changing magnetic flux in the loop; the direction of this flow will be such that the magnetic force on the current will oppose the falling motion. The loop will fall freely until it starts to enter the magnetic field. It will then slow down very suddenly — falling but not freely. When it is completely inside the region, it will again fall freely since the flux through the loop no longer changes. As it starts to leave the region, it will again fall slowly; it will then fall freely once it completely leaves the region.

(c) Is $\vec{B} = B_0 z\hat{z}$ a possible magnetic field? If so, justify your statement. If not, fix it.
It is NOT a valid field: Notice that $\vec{\nabla} \cdot \vec{B} \neq 0$! There are lots of ways to fix it. The simplest is probably to include a component along the x or y directions — e.g., put $\vec{B} = B_0\,(z\hat{z} - x\hat{x})$.

PROBLEM 2 (35 pts)

In this problem, you will look at a circuit that contains a constant *current* source: the total current which comes out of this device is **always** I_0 no matter what EMF is required. The switch S is closed at $t = 0$.
First, this source is hooked up to a pair of resistors in parallel:



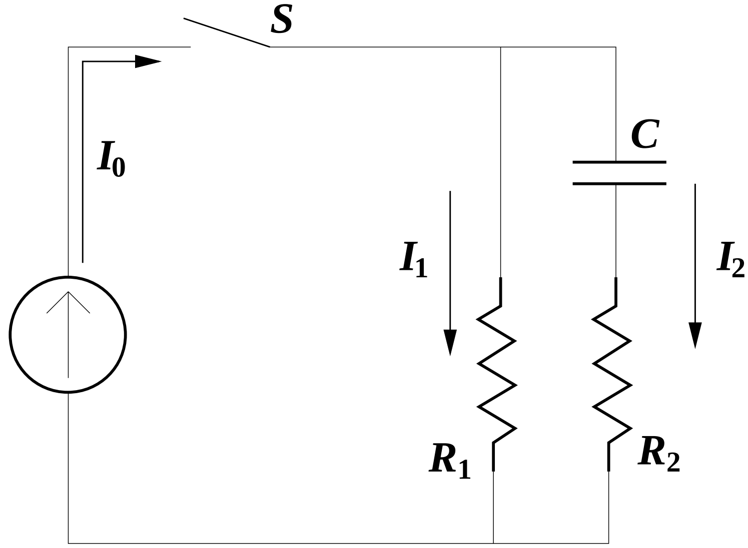
(a) Find the EMF $\mathcal{E}$ produced by the constant current source, as well as the currents I_1 (flowing through resistor R_1) and I_2 (flowing through R_2).
The resistors in parallel have an equivalent resistance given by $1/R_{\rm eq} = 1/R_1 + 1/R_2$, or $R_{\rm eq} = R_1R_2/(R_1 + R_2)$. The EMF produced by the current source is thus

$$\mathcal{E} = \frac{I_0 R_1 R_2}{R_1 + R_2} .$$

The currents through the two resistors are given by

$$\begin{aligned} I_1 &= \frac{\mathcal{E}}{R_1} = \frac{I_0 R_2}{R_1 + R_2} \\ I_2 &= \frac{\mathcal{E}}{R_2} = \frac{I_0 R_1}{R_1 + R_2} . \end{aligned}$$

The circuit is now modified: a capacitor and a switch are added. The capacitor is initially uncharged.



The switch is closed at $t = 0$.

(b) Find the initial currents $I_1(t = 0)$, $I_2(t = 0)$, and the late time currents, $I_1(t \to \infty)$, $I_2(t \to \infty)$. You should be able to do this with very little calculation. (Express your answers in terms of I_0 and the parameters of the circuit.)
At $t = 0$, there is no charge on the capacitor and it behaves identically to the circuit in part (a). At late times, the capacitor is fully charged and no current flows through R_2.

$$I_1(t=0) = \frac{I_0 R_2}{R_1 + R_2} \,, \qquad I_2(t=0) = \frac{I_0 R_1}{R_1 + R_2}$$
$$I_1(t \to \infty) = I_0 \,, \qquad I_2(t \to \infty) = 0 \,.$$

(c) Use Kirchhoff's laws to write down two equations relating $I_1(t)$, $I_2(t)$, I_0, and the charge on the capacitor Q. Write down a third equation relating Q to I_2.

$$I_0 = I_1 + I_2 \,, \qquad I_1 R_1 - I_2 R_2 - \frac{Q}{C} = 0 \,, \qquad I_2 = \frac{dQ}{dt} \,.$$

(d) Using the results of (c), find the late time charge, $Q(t \to \infty)$. (Express your answer in terms of I_0 and the parameters of the circuit.)
For $t \to \infty$ we have $I_2 = 0$, $I_1 = I_0$. Plugging into Kirchhoff and solving for Q, we find

$$I_0 R_1 - \frac{Q}{C} = 0$$
$$\longrightarrow Q = I_0 R_1 C \,.$$

(e) Assume $I_1(t) = A + Be^{-t/T_1}$, $I_2(t) = D + Ee^{-t/T_2}$, $Q = F + Ge^{-t/T_3}$. Using your results from parts (b) and (c), find A, B, D, E, F, and G.

$$I_1(0) = A + B = \frac{I_0 R_2}{R_1 + R_2} \; ; \qquad I_1(\infty) = A = I_0$$
$$\longrightarrow A = I_0 \; , \qquad B = -\frac{I_0 R_1}{R_1 + R_2}$$

$$I_2(0) = D + E = \frac{I_0 R_1}{R_1 + R_2} \; ; \qquad I_2(\infty) = D = 0$$
$$\longrightarrow D = 0 \; , \qquad E = \frac{I_0 R_1}{R_1 + R_2}$$

$$Q(0) = F + G = 0 \; ; \qquad Q(\infty) = F = C I_0 R_1$$
$$\longrightarrow F = C I_0 R_1 \; , \qquad G = -C I_0 R_1$$

(f) Using your answers to parts (c) and (e), find T_1, T_2, and T_3.
Let's begin with $I_2 = dQ/dt$:

$$I_2 = \frac{I_0 R_1}{R_1 + R_2} e^{-t/T_2} \; ; \qquad \frac{dQ}{dt} = \frac{C I_0 R_1}{T_3} e^{-t/T_3}$$

These are equal at $t = 0$ if

$$T_3 = C(R_1 + R_2)$$

These are equal for ALL t if $T_2 = T_3$. Plugging into the remaining Kirchhoff equation, we find that everything works IF we also choose $T_1 = T_3$. So,

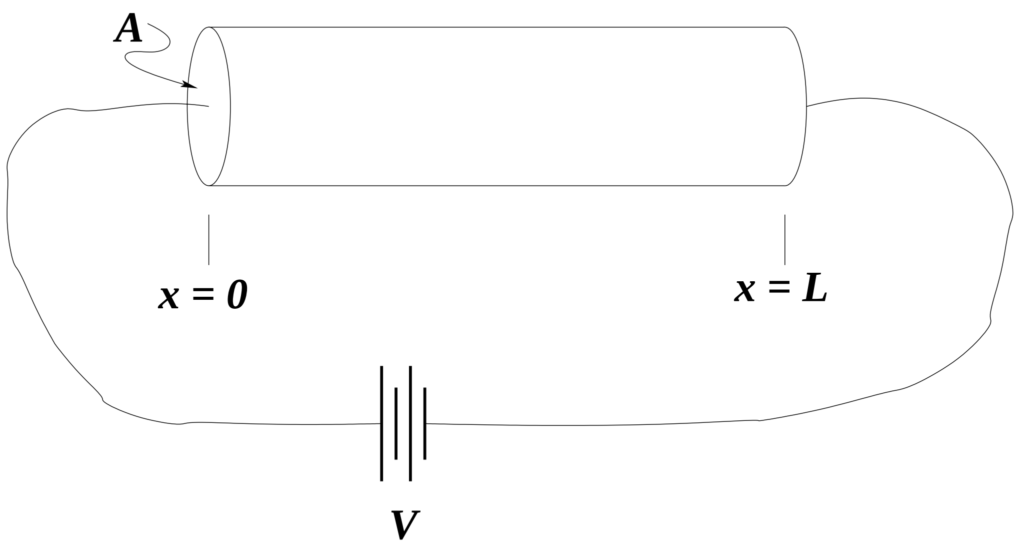
$$T_1 = T_2 = T_3 = C(R_1 + R_2) \; .$$

(g) After a long time, the switch is opened. Find the total energy dissipated in the resistors after the switch is opened. Hint: this can be done without a single integral!
It's the energy stored in the capacitor:

$$U_{\text{diss}} = \frac{Q(\infty)^2}{2C} = \frac{1}{2} C I_0^2 R_1^2 \; .$$

PROBLEM 3 (20 pts)

The segment of wire shown below is oriented along the x axis, has a total length L and a cross-sectional area A. It is made of material whose conductivity varies along the wire as $\sigma_c(x) = \sigma_0(L/x)$. This segment of wire is connected to a battery using zero-resistance cables. The battery puts out an EMF V.



In what follows, assume that everything is in a steady state ($\partial\rho/\partial t = 0$), and that the wires bring current uniformly to and from the ends of the segment. $\hat{x}$ points from $x = 0$ to $x = L$.

(a) What is the resistance R of the wire segment?
Treat the segment as a sequence of resistive slabs of length dx. Each slab has resistance

$$dR = \frac{1}{\sigma_c(x)}\frac{dx}{A} = \frac{1}{\sigma_0 LA} x\, dx$$

To get the total resistance, we add the slabs in series — i.e., we integrate:

$$R = \int_0^L dR(x) = \frac{1}{\sigma_0 LA}\int_0^L x\, dx = \frac{L}{2\sigma_0 A} \ .$$

(b) What is the current density $\vec{J}$ that flows in the wire? Include direction.
The current is found by Ohm's law, $V = IR$, and flows in the positive x direction. The magnitude of the current density is $J = I/A$. Putting all this this together, we have

$$\vec{J} = \frac{V}{RA}\hat{x} = \frac{2\sigma_0 V}{L}\hat{x} \ .$$

(c) What is the electric field $\vec{E}$ in the wire? Include direction.
The other form of Ohm's law tells us that $\vec{J} = \sigma_c \vec{E}$, so

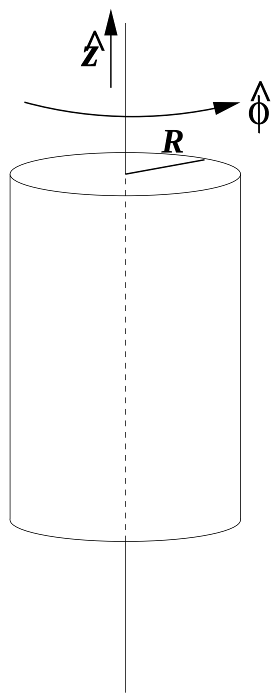
$$\vec{E} = \frac{1}{\sigma_c}\vec{J} = \frac{2Vx}{L^2}\hat{x} \ .$$

(d) What is the charge density ρ in the wire?

$$\rho = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi}\frac{\partial E_x}{\partial x} = \frac{V}{2\pi L^2}$$

PROBLEM 4 (25 pts)

Consider a solid, conducting infinitely long cylinder, of radius R, oriented parallel to the z-axis. The unit vector $\hat{z}$ points up; the unit vector $\hat{\phi}$ (cylindrical coordinates) points "around". $\hat{z}$ and $\hat{\phi}$ are related by right-hand rule: point your right-hand thumb along $\hat{z}$, and your fingers curl in the direction of $\hat{\phi}$. $\hat{r}$ (not shown) points radially out from the z-axis.



There is a vector potential $\vec{A} = A_0(r/R)\hat{z}$ inside this cylinder (for $r < R$). Note that $A_0 > 0$.

(a) Compute the magnetic field $\vec{B}$ inside the cylinder ($r < R$). (The formula sheet may help you here.)

$$\vec{B} = \textbf{curl}\,\vec{A} = -\frac{\partial A_z}{\partial r}\hat{\phi} = -\frac{A_0}{R}\hat{\phi}$$

(b) Compute the total current I flowing in the cylinder. Does it flow up or down?
Use Faraday's law on a contour just at the surface of the cylinder:

$$\oint \vec{B}\cdot d\vec{s} = \frac{4\pi I}{c}$$

$$2\pi R\frac{A_0}{R} = \frac{4\pi I}{c} \longrightarrow I = \frac{A_0 c}{2}\ .$$

This current flows DOWN (in order to produce a magnetic field pointing in the $-\hat{\phi}$ direction).

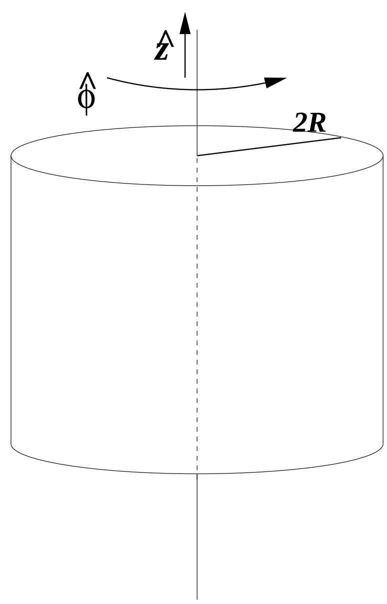
(c) Compute the magnetic field for $r > R$.
Use Faraday's law for a circular contour at $r > R$:

$$\oint \vec{B} \cdot d\vec{s} = \frac{4\pi I}{c}$$
$$2\pi r B(r) = \frac{4\pi I}{c}$$

Now, plug in $I = A_0 c/2$, and use right-hand rule to get direction:

$$\vec{B} = -\frac{A_0}{r}\hat{\phi}\ .$$

Now, replace this cylinder with one of radius $2R$.



A current density $\vec{J} = J_0 (r/R)^2 \hat{z}$ flows in this cylinder. J_0 may be positive or negative.

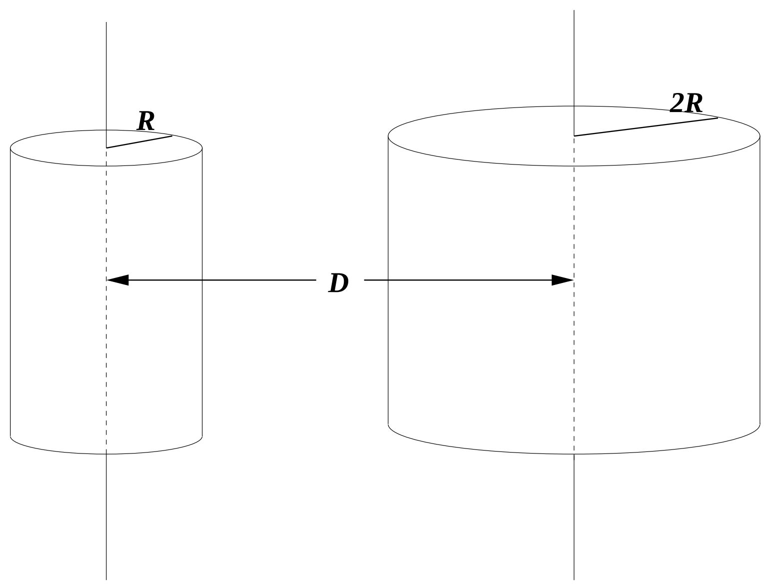
(d) Write down the magnetic field $\vec{B}$ for all r that arises from this current.
More fun with Faraday's law. For $r \leq 2R$, we have

$$\begin{aligned}
2\pi r B(r) &= \frac{4\pi}{c}\int_0^r \vec{J}(r)\cdot d\vec{a} \\
&= \frac{8\pi^2}{c}\frac{J_0}{R^2}\int_0^r r^3\,dr \\
&= \frac{2\pi^2}{c}\frac{J_0\,r^4}{R^2} \\
\longrightarrow \vec{B} &= \frac{\pi}{c}\frac{J_0\,r^3}{R^2}\hat{\phi}\ .
\end{aligned}$$

For $r \geq 2R$, we have

$$\begin{aligned}
2\pi r B(r) &= \frac{4\pi}{c}\int_0^{2R} \vec{J}(r)\cdot d\vec{a} \\
&= \frac{8\pi^2}{c}\frac{J_0}{R^2}\int_0^{2R} r^3\, dr \\
&= \frac{2\pi^2}{c}\frac{J_0\,(2R)^4}{R^2} = \frac{32\pi^2}{c} J_0\, R^2 \\
\longrightarrow \vec{B} &= \frac{16\pi}{c}\frac{J_0\, R^2}{r}\hat{\phi}\,.
\end{aligned}$$

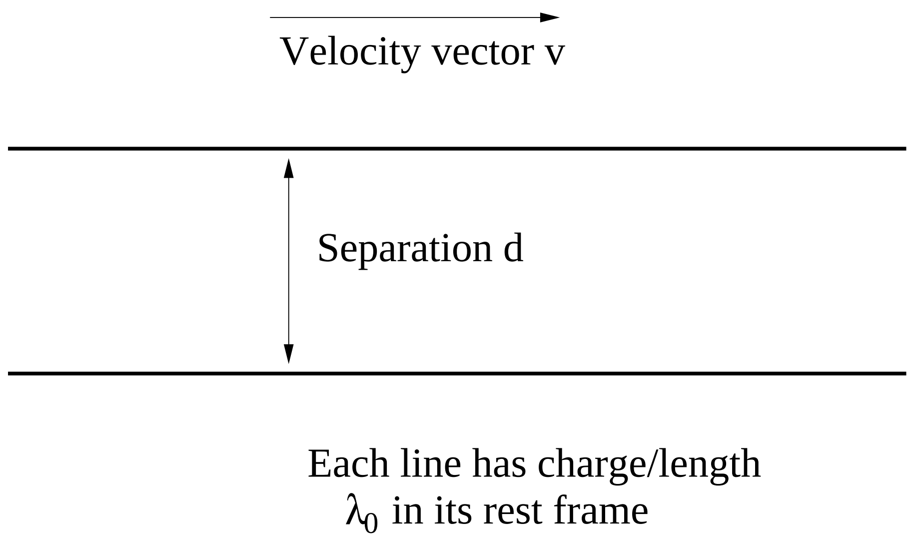
Both cylinders are now placed next to one another, separated by a distance $D > 4R$ (so that $D/2 > 2R$ is outside both cylinders).



(e) Find the value of J_0 such that the total magnetic field vanishes midway between them (i.e., at a distance $r = D/2$ from the centers of both cylinders).
Let out of the page be the positive direction. Then,

$$\begin{aligned}
\vec{B}_{\text{left}} + \vec{B}_{\text{right}} &= 0 \\
\frac{A_0}{D/2} + \frac{16\pi}{c}\frac{J_0\, R^2}{D/2} &= 0 \\
\longrightarrow J_0 &= -\frac{cA_0}{16\pi R^2}
\end{aligned}$$

PROBLEM 5 (11 pts)



Two infinite lines of charge with charge per unit length λ_0 in their rest frame are separated by a distance d. These charges are moving in a direction parallel to their length with speed v. This speed could be close to the speed of light!

(a) In the rest frame, what is the electric force per unit length that the top line feels due to the bottom line? Give direction (up or down) and magnitude.
The bottom line generates an electric field of magnitude

$$E = \frac{2\lambda_0}{d}$$

at the top line. The force per unit length is thus

$$F^E_{\text{rest}}/L = \frac{2\lambda_0^2}{d} \ .$$

The force is repulsive, so the direction is up.

(b) In the lab frame, what is the electric force per unit length that the top line feels due to the bottom line? Give direction (up or down) and magnitude.
Same answer ... but now all lengths contract, so the density is increased by a factor of γ:

$$F^E_{\text{lab}}/L = \frac{2(\gamma\lambda_0)^2}{d} = \frac{2\gamma^2\lambda_0^2}{d} \ .$$

The direction is up.

(c) In the lab frame, what is the magnetic force per unit length that the top line feels due to the bottom line? Give direction (up or down) and magnitude.
The moving line charge generates a current of magnitude $I = \gamma\lambda_0 v$, which generates a magnetic field

$$B = \frac{2\gamma\lambda_0 v}{cd}$$

The top line charge generates the same current, so there is a force per unit length $\vec{I} \times \vec{B}/c$:

$$F^{B}_{\text{lab}}/L = \frac{2(\gamma\lambda_0 v)^2}{c^2 d} = \frac{2\gamma^2 v^2 \lambda_0^2}{c^2 d}\ .$$

The direction is down.

(d) What is the total force per unit length in the lab frame?

$$F^{\text{TOT}}_{\text{lab}}/L = F^{E}_{\text{lab}}/L + F^{B}_{\text{lab}}/L = \frac{2\gamma^2\lambda_0^2}{d}\left(1 - v^2/c^2\right) = \frac{2\lambda_0^2}{d}$$

The net direction is up. This is identical to the force per unit length in the rest frame!