6.3 Circular Motion: Tangential and Radial Acceleration

When the motion of an object is described in polar coordinates, the acceleration has two components, the tangential component a_θ, and the radial component, a_r. We can write the acceleration vector as

$$\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}}(t) + a_\theta\,\hat{\boldsymbol{\theta}}(t)\,. \tag{6.3.1}$$

Keep in mind that as the object moves in a circle, the unit vectors $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ change direction and hence are not constant in time.

We will begin by calculating the tangential component of the acceleration for circular motion. Suppose that the tangential velocity $v_\theta = r\,d\theta/dt$ is changing in magnitude due to the presence of some tangential force; we shall now consider that $d\theta/dt$ is changing in time, (the magnitude of the velocity is changing in time). Recall that in polar coordinates the velocity vector Eq. (6.2.8) can be written as

$$\vec{\mathbf{v}}(t) = r\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}(t)\,. \tag{6.3.2}$$

We now use the product rule to determine the acceleration.

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}(t)}{dt} = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\frac{d\hat{\boldsymbol{\theta}}(t)}{dt}\,. \tag{6.3.3}$$

Recall from Eq. (6.2.3) that $\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}$. So we can rewrite Eq. (6.3.3) as

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\frac{d}{dt}(-\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}})\,. \tag{6.3.4}$$

We again use the chain rule (Eqs. (6.2.5) and (6.2.6)) and find that

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta(t)}{dt}\left(-\cos\theta(t)\frac{d\theta(t)}{dt}\hat{\mathbf{i}} - \sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}}\right). \tag{6.3.5}$$

Recall that $\omega \equiv d\theta / dt$, and from Eq. (6.2.2), $\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}$, therefore the acceleration becomes

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta(t)}{dt^2}\,\hat{\boldsymbol{\theta}}(t) - r\left(\frac{d\theta(t)}{dt}\right)^2 \hat{\mathbf{r}}(t)\,. \tag{6.3.6}$$

The *tangential component of the acceleration* is then

$$a_\theta = r\frac{d^2\theta(t)}{dt^2}\,. \tag{6.3.7}$$

The *radial component of the acceleration* is given by

$$a_r = -r\left(\frac{d\theta(t)}{dt}\right)^2 = -r\omega^2 < 0\ . \tag{6.3.8}$$

Because $a_r < 0$, that radial vector component $\vec{\mathbf{a}}_r(t) = -r\omega^2\,\hat{\mathbf{r}}(t)$ is always directed towards the center of the circular orbit.

Example 6.1 Circular Motion Kinematics

A particle is moving in a circle of radius R. At $t = 0$, it is located on the x-axis. The angle the particle makes with the positive x-axis is given by $\theta(t) = At^3 - Bt$, where A and B are positive constants. Determine (a) the velocity vector, and (b) the acceleration vector. Express your answer in polar coordinates. At what time is the centripetal acceleration zero?

Solution:

The derivatives of the angle function $\theta(t) = At^3 - Bt$ are $d\theta / dt = 3At^2 - B$ and $d^2\theta / dt^2 = 6At$. Therefore the velocity vector is given by

$$\vec{\mathbf{v}}(t) = R\frac{d\theta(t)}{dt}\,\hat{\boldsymbol{\theta}}(t) = R(3At^2 - Bt)\hat{\boldsymbol{\theta}}(t)\,.$$

The acceleration is given by

$$\begin{aligned}\vec{\mathbf{a}}(t) &= R\frac{d^2\theta(t)}{dt^2}\,\hat{\boldsymbol{\theta}}(t) - R\left(\frac{d\theta(t)}{dt}\right)^2 \hat{\mathbf{r}}(t) \\ &= R(6At)\,\hat{\boldsymbol{\theta}}(t) - R\left(3At^2 - B\right)^2 \hat{\mathbf{r}}(t)\end{aligned}.$$

The centripetal acceleration is zero at time $t = t_1$ when

$$3At_1^2 - B = 0 \Rightarrow t_1 = \sqrt{B/3A}\,.$$