4.6 One Dimensional Kinematics and Integration

When the acceleration $a(t)$ of an object is a non-constant function of time, we would like to determine the time dependence of the position function $x(t)$ and the x-component of the velocity $v(t)$. Because the acceleration is non-constant we no longer can use Eqs. (4.4.2) and (4.4.9). Instead we shall use integration techniques to determine these functions.

4.6.1 Change of Velocity as the Indefinite Integral of Acceleration

Consider a time interval $t_1 < t < t_2$. Recall that by definition the derivative of the velocity $v(t)$ is equal to the acceleration $a(t)$,

$$\frac{dv(t)}{dt} = a(t) \ . \tag{4.5.1}$$

Integration is defined as the inverse operation of differentiation or the 'anti-derivative'. For our example, the function $v(t)$ is called the ***indefinite integral*** of $a(t)$ with respect to t, and is unique up to an additive constant C. We denote this by writing

$$v(t) + C = \int a(t)\,dt \ . \tag{4.5.2}$$

The symbol $\int ...dt$ means the 'integral, with respect to t, of ...", and is thought of as the inverse of the symbol $\dfrac{d}{dt}.....$. Equivalently we can write the differential $dv(t) = a(t)dt$, called the ***integrand***, and then Eq. (4.5.2) can be written as

$$v(t) + C = \int dv(t) \ , \tag{4.5.3}$$

which we interpret by saying that the integral of the differential of function is equal to the function plus a constant.

Example 4.6 Non-constant acceleration

Suppose an object at time $t = 0$ has initial non-zero velocity v_0 and acceleration $a(t) = bt^2$, where b is a constant. Then $dv(t) = bt^2 dt = d(bt^3 / 3)$. The velocity is then $v(t) + C = \int d(bt^3 / 3) = bt^3 / 3$. At $t = 0$, we have that $v_0 + C = 0$. Therefore $C = -v_0$ and the velocity as a function of time is then $v(t) = v_0 + (bt^3 / 3)$.

4.6.2 Area as the Indefinite Integral of Acceleration

Consider the graph of a positive-valued acceleration function $a(t)$ vs. t for the interval $t_1 \leq t \leq t_2$, shown in Figure 4.14a. Denote the area under the graph of $a(t)$ over the interval $t_1 \leq t \leq t_2$ by $A_{t_1}^{t_2}$.
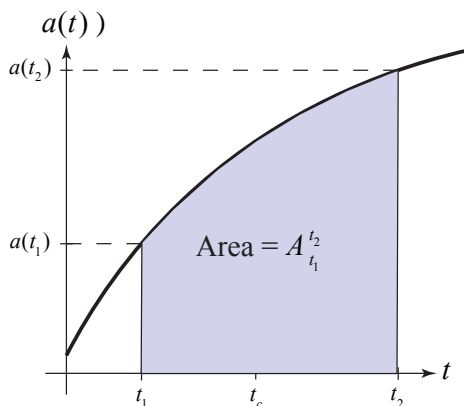

Figure 4.14a: Area under the graph of acceleration over an interval $t_1 \leq t \leq t_2$
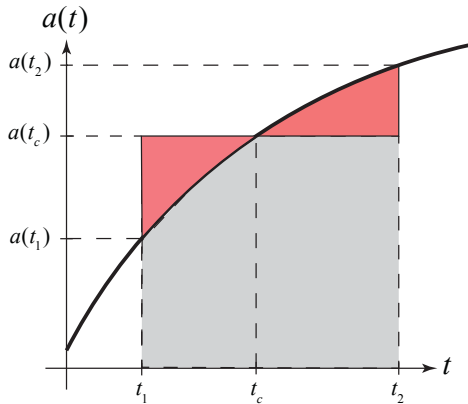
a(t)

Figure 4.14b: Intermediate value Theorem. The shaded regions above and below the curve have equal areas.

The ***Intermediate Value Theorem*** states that there is at least one time t_c such that the area $A_{t_1}^{t_2}$ is equal to

$$A_{t_1}^{t_2} = a(t_c)(t_2 - t_1) \ .$$

(4.5.4)

In Figure 4.14b, the shaded regions above and below the curve have equal areas, and hence the area $A_{t_1}^{t_2}$ under the curve is equal to the area of the rectangle given by $a(t_c)(t_2 - t_1)$.
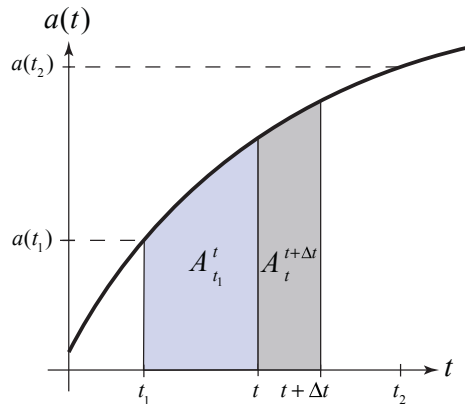

Figure 4.15 Area function is additive

We shall now show that the derivative of the area function is equal to the acceleration and thererfore we can write the area function as an indefinite integral. From Figure 4.15, the area function satisfies the condition that

$$A_{t_1}^{t} + A_{t}^{t+\Delta t} = A_{t_1}^{t+\Delta t} \ .$$

(4.5.5)

Let the small increment of area be denoted by $\Delta A_{t_1}^{t} = A_{t_1}^{t+\Delta t} - A_{t_1}^{t} = A_{t}^{t+\Delta t}$. By the Intermediate Value Theorem

$$\Delta A\,{}^{t}_{t_1} = a(t_c)\Delta t \ , \tag{4.5.6}$$

where $t \le t_c \le t + \Delta t$. In the limit as $\Delta t \to 0$,

$$\frac{dA\,{}^{t}_{t_1}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A\,{}^{t}_{t_1}}{\Delta t} = \lim_{t_c \to t} a(t_c) = a(t) \ , \tag{4.5.7}$$

with the initial condition that when $t = t_1$, the area $A\,{}^{t_1}_{t_1} = 0$ is zero. Because $v(t)$ is also an integral of $a(t)$, we have that

$$A\,{}^{t}_{t_1} = \int a(t)\,dt = v(t) + C \ . \tag{4.5.8}$$

When $t = t_1$, the area $A\,{}^{t_1}_{t_1} = 0$ is zero, therefore $v(t_1) + C = 0$, and so $C = -v(t_1)$. Therefore Eq. (4.5.8) becomes

$$A\,{}^{t}_{t_1} = v(t) - v(t_1) = \int a(t)\,dt \ . \tag{4.5.9}$$

When we set $t = t_2$, Eq. (4.5.9) becomes

$$A\,{}^{t_2}_{t_1} = v(t_2) - v(t_1) = \int a(t)\,dt \ . \tag{4.5.10}$$

The area under the graph of the positive-valued acceleration function for the interval $t_1 \le t \le t_2$ can be found by integrating $a(t)$.

4.6.3 Change of Velocity as the Definite Integral of Acceleration

Let $a(t)$ be the acceleration function over the interval $t_i \le t \le t_f$. Recall that the velocity $v(t)$ is an integral of $a(t)$ because $dv(t)/dt = a(t)$. Divide the time interval $[t_i, t_f]$ into n equal time subintervals $\Delta t = (t_f - t_i)/n$. For each subinterval $[t_j, t_{j+1}]$, where the index $j = 1,\ 2,\ ...,n$, $t_1 = t_i$ and $t_{n+1} = t_f$, let t_{c_j} be a time such that $t_j \le t_{c_j} \le t_{j+1}$. Let

$$S_n = \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t \ . \tag{4.5.11}$$

S_n is the sum of the blue rectangle shown in Figure 4.16a for the case $n = 4$. The **Fundamental Theorem of Calculus** states that in the limit as $n \to \infty$, the sum is equal to the change in the velocity during the interval $[t_i, t_f]$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t = v(t_f) - v(t_i)\ .\qquad (4.5.12)$$
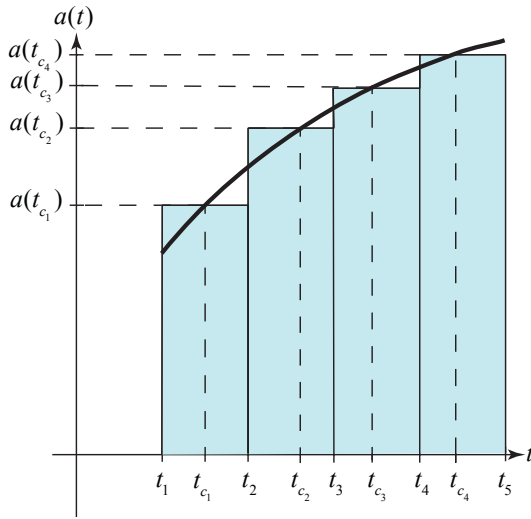

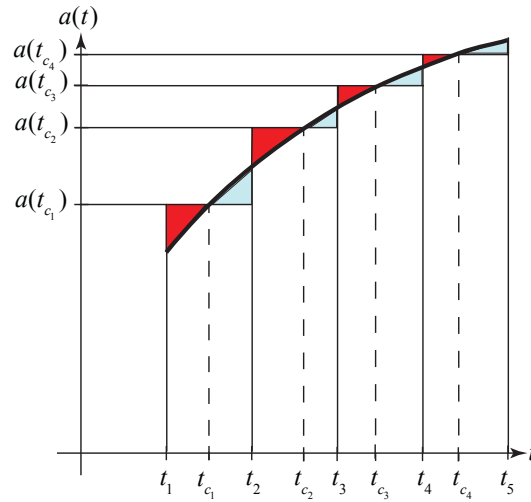**Figure 4.16a** Graph of $a(t)$ vs. t **Figure 4.16b** Graph of $a(t)$ vs. t

The limit of the sum in Eq. (4.5.12) is a number, which we denote by the symbol

$$\int_{t_i}^{t_f} a(t)\, dt \equiv \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t = v(t_f) - v(t_i)\ ,\qquad (4.5.13)$$

and is called the ***definite integral*** of $a(t)$ from t_i to t_f. The times t_i and t_f are called the limits of integration, t_i the lower limit and t_f the upper limit. The definite integral is a linear map that takes a function $a(t)$ defined over the interval $[t_i, t_f]$ and gives a number. The map is linear because

$$\int_{t_i}^{t_f} (a_1(t) + a_2(t))\, dt = \int_{t_i}^{t_f} a_1(t)\, dt + \int_{t_i}^{t_f} a_2(t)\, dt\ ,\qquad (4.5.14)$$

Suppose the times t_{c_j}, $j = 1,...,n$, are selected such that each t_{c_j} satisfies the Intermediate Value Theorem,

$$\Delta v_j \equiv v(t_{j+1}) - v(t_j) = \frac{dv(t_{c_j})}{dt}\,\Delta t = a(t_{c_j})\Delta t\ ,\qquad (4.5.15)$$

where $a(t_{c_j})$ is the instantaneous acceleration at t_{c_j}, (Figure 4.16b). Then the sum of the changes in the velocity for the interval $[t_i, t_f]$ is

$$\sum_{j=1}^{j=n} \Delta v_j = (v(t_2) - v(t_1)) + (v(t_3) - v(t_2)) + \cdots + (v(t_{n+1}) - v(t_n)) = v(t_{n+1}) - v(t_1)$$

$$= v(t_f) - v(t_i).$$

(4.5.16)

where $v(t_f) = v(t_{n+1})$ and $v(t_i) = v(t_1)$. Substituting Eq. (4.5.15) into Equation (4.5.16) yields the exact result that the change in the x-component of the velocity is give by this finite sum.

$$v(t_f) - v(t_i) = \sum_{j=1}^{j=n} \Delta v_j = \sum_{j=1}^{j=n} a(t_{c_j}) \Delta t .$$

(4.5.17)

We do not specifically know the intermediate values $a(t_{c_j})$ and so Eq. (4.5.17) is not useful as a calculating tool. The statement of the Fundamental Theorem of Calculus is that the limit as $n \rightarrow \infty$ of the sum in Eq. (4.5.12) is independent of the choice of the set of t_{c_j}. Therefore the exact result in Eq. (4.5.17) is the limit of the sum.

Thus we can evaluate the definite integral if we know any indefinite integral of the integrand $a(t)dt = dv(t)$.

Additionally, provided the acceleration function has only non-negative values, the limit is also equal to the area under the graph of $a(t)$ vs. t for the time interval, $[t_i, t_f]$:

$$A_{t_i}^{t_f} = \int_{t_i}^{t_f} a(t)\, dt .$$

(4.5.18)

In Figure 4.14, the red areas are an overestimate and the blue areas are an underestimate. As $N \rightarrow \infty$, the sum of the red areas and the sum of the blue areas both approach zero. If there are intervals in which $a(t)$ has negative values, then the summation is a sum of signed areas, positive area above the t-axis and negative area below the t-axis.

We can determine both the change in velocity for the time interval $[t_i, t_f]$ and the area under the graph of $a(t)$ vs. t for $[t_i, t_f]$ by integration techniques instead of limiting arguments. We can turn the linear map into a function of time, instead of just giving a number, by setting $t_f = t$. In that case, Eq. (4.5.13) becomes

$$v(t) - v(t_i) = \int_{t'=t_i}^{t'=t} a(t')dt' .$$ (4.5.19)

Because the upper limit of the integral, $t_f = t$, is now treated as a variable, we shall use the symbol t' as the integration variable instead of t.

4.6.4 Displacement as the Definite Integral of Velocity

We can repeat the same argument for the definite integral of the x-component of the velocity $v(t)$ *vs.* time t. Because $x(t)$ is an integral of $v(t)$ the definite integral of $v(t)$ for the time interval $[t_i, t_f]$ is the displacement

$$x(t_f) - x(t_i) = \int_{t'=t_i}^{t'=t_f} v(t')dt' .$$ (4.5.20)

If we set $t_f = t$, then the definite integral gives us the position as a function of time

$$x(t) = x(t_i) + \int_{t'=t_i}^{t'=t} v(t')dt' .$$ (4.5.21)

Summarizing the results of these last two sections, for a given acceleration $a(t)$, we can use integration techniques, to determine the change in velocity and change in position for an interval $[t_i, t]$, and given initial conditions (x_i, v_i), we can determine the position $x(t)$ and the x-component of the velocity $v(t)$ as functions of time.

Example 4.5 Non-constant Acceleration

Let's consider a case in which the acceleration, $a(t)$, is not constant in time,

$$a(t) = b_0 + b_1 t + b_2 t^2 .$$ (4.5.22)

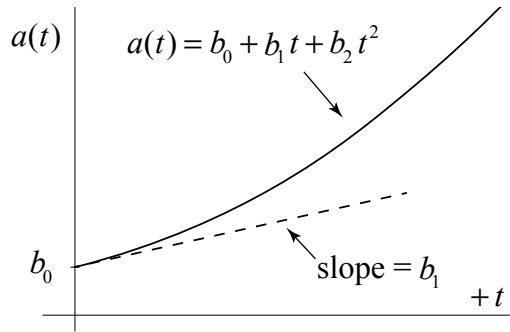The graph of the x-component of the acceleration *vs.* time is shown in Figure 4.16

Figure 4.16 Non-constant acceleration *vs.* time graph.

Denote the initial velocity at $t = 0$ by v_0. Then, the change in the x-component of the velocity as a function of time can be found by integration:

$$v(t) - v_0 = \int_{t'=0}^{t'=t} a(t')\,dt' = \int_{t'=0}^{t'=t} (b_0 + b_1 t' + b_2 t'^2)\,dt' = b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \qquad (4.5.23)$$

The x-component of the velocity as a function in time is then

$$v(t) = v_0 + b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \qquad (4.5.24)$$

Denote the initial position at $t = 0$ by x_0. The displacement as a function of time is

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v(t')\,dt'. \qquad (4.5.25)$$

Use Equation (4.5.27) for the x-component of the velocity in Equation (4.5.24) and then integrate to determine the displacement as a function of time:

$$
\begin{aligned}
x(t) - x_0 &= \int_{t'=0}^{t'=t} v(t')\,dt' \\
&= \int_{t'=0}^{t'=t} \left(v_0 + b_0 t' + \frac{b_1 t'^2}{2} + \frac{b_2 t'^3}{3} \right) dt' = v_0 t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}.
\end{aligned}
\qquad (4.5.26)
$$

Finally the position as a function of time is then

$$x(t) = x_0 + v_{x,0}\, t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \qquad (4.5.27)$$

Example 4.6 Bicycle and Car

A car is driving through a green light at $t = 0$ located at $x = 0$ with an initial speed $v_{c,0} = 12 \text{ m} \cdot \text{s}^{-1}$. At time $t_1 = 1 \text{s}$, the car starts braking until it comes to rest at time t_2. The acceleration of the car as a function of time is given by the piecewise function

$$a_c(t) = \begin{cases} 0; & 0 < t < t_1 = 1 \text{ s} \\ b(t - t_1); & 1 \text{ s} < t < t_2 \end{cases},$$

where $b = -(6 \text{ m} \cdot \text{s}^{-3})$.

(a) Find the x-component of the velocity and the position of the car as a function of time.
(b) A bicycle rider is riding at a constant speed of $v_{b,0}$ and at $t = 0$ is 17 m behind the car. The bicyclist reaches the car when the car just comes to rest. Find the speed of the bicycle.

Solution: a) In order to apply Eq. (4.5.19), we shall treat each stage separately. For the time interval $0 < t < t_1$, the acceleration is zero so the x-component of the velocity is constant. For the second time interval $t_1 < t < t_2$, the definite integral becomes

$$v_c(t) - v_c(t_1) = \int_{t'=t_1}^{t'=t} b(t' - t_1) \, dt'$$

Because $v_c(t_1) = v_{c0}$, the x-component of the velocity is then

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t \le t_1 \\ v_{c0} + \displaystyle\int_{t'=t_1}^{t'=t} b(t' - t_1) \, dt'; & t_1 \le t < t_2 \end{cases}.$$

Integrate and substitute the two endpoints of the definite integral, yields

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t \le t_1 \\ v_{c0} + \dfrac{1}{2} b(t - t_1)^2; & t_1 \le t < t_2 \end{cases}.$$

In order to use Eq. (4.5.25), we need to separate the definite integral into two integrals corresponding to the two stages of motion, using the correct expression for the velocity for each integral. The position function is then

$$x_c(t) = \begin{cases} x_{c0} + \int\limits_{t'=0}^{t'=t_1} v_{c0}\, dt'; & 0 < t \le t_1 \\[2mm] x_c(t_1) + \int\limits_{t'=t_1}^{t'=t} \left(v_{c0} + \frac{1}{2}b(t'-t_1)^2 \right) dt; & t_1 \le t < t_2 \end{cases}$$

Upon integration we have

$$x_c(t) = \begin{cases} x_c(0) + v_{c0}t; & 0 < t \le t_1 \\[2mm] x_c(t_1) + \left(v_{c0}(t'-t_1) + \frac{1}{6}b(t'-t_1)^3 \right)\Big|_{t'=t_1}^{t'=t}; & t_1 \le t < t_2 \end{cases}$$

We chose our coordinate system such that the initial position of the car was at the origin, $x_{c0} = 0$, therefore $x_c(t_1) = v_{c0}t_1$. So after substituting in the endpoints of the integration interval we have that

$$x_c(t) = \begin{cases} v_{c0}t; & 0 < t \le t_1 \\[2mm] v_{c0}t_1 + v_{c0}(t-t_1) + \frac{1}{6}b(t-t_1)^3; & t_1 \le t < t_2 \end{cases}$$

(b) We are looking for the instant t_2 that the car has come to rest. So we use our expression for the x-component of the velocity the interval $t_1 \le t < t_2$, where we set $t = t_2$ and $v_c(t_2) = 0$:

$$0 = v_c(t_2) = v_{c0} + \frac{1}{2}b(t_2 - t_1)^2.$$

Solving for t_2 yields

$$t_2 = t_1 + \sqrt{-\frac{2v_{c0}}{b}},$$

where we have taken the positive square root. Substitute the given values then yields

$$t_2 = 1\,\text{s} + \sqrt{-\frac{2(12\,\text{m}\cdot\text{s}^{-1})}{(-6\,\text{m}\cdot\text{s}^{-3})}} = 3\,\text{s}.$$

The position of the car at t_2 is then given by

$$x_c(t_2) = v_{c0} t_1 + v_{c0}(t_2 - t_1) + \frac{1}{6} b(t_2 - t_1)^3$$

$$x_c(t_2) = v_{c0} t_1 + v_{c0}\sqrt{-2v_{c0}/b} + \frac{1}{6} b(-2v_{c0}/b)^{3/2}$$

$$x_c(t_2) = v_{c0} t_1 + \frac{2\sqrt{2}(v_{c0}^{3/2})}{3(-b)^{1/2}}$$

where we used the condition that $t_2 - t_1 = \sqrt{-2v_{c0}/b}$. Substitute the given values then yields

$$x_c(t_2) = v_{c0} t_1 + 2\frac{4\sqrt{2}(v_{c0})^{3/2}}{3(-b)^{1/2}} = (12 \text{ m} \cdot \text{s}^{-1})(1 \text{ s}) + \frac{4\sqrt{2}((12 \text{ m} \cdot \text{s}^{-1})^{3/2}}{3((6 \text{ m} \cdot \text{s}^{-3}))^{1/2}} = 28 \text{ m} .$$

b) Because the bicycle is traveling at a constant speed with an initial position $x_{b0} = -17 \text{ m}$, the position of the bicycle is given by $x_b(t) = -17 \text{ m} + v_b t$. The bicycle and car intersect at time $t_2 = 3 \text{ s}$, where $x_b(t_2) = x_c(t_2)$. Therefore $-17 \text{ m} + v_b(3 \text{ s}) = 28 \text{ m}$. So the speed of the bicycle is $v_b = 15 \text{ m} \cdot \text{s}^{-1}$.

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