1.0 Review of Electromagnetic Field Theory

Selected aspects of electromagnetic theory are reviewed in this section, with emphasis on concepts which are useful in understanding magnet design. Detailed, rigorous treatments are presented in standard texts on the subject. [1,2,3]

1.1 General Form of the Equations

The concepts of electric and magnetic fields are related to the observation of forces experienced by an electric charge. A charge of q coulombs moving with velocity **v** can experience a force independent of its velocity and perpendicular to it. The total is the Lorentz force $\vec{F}$ which can be expressed as

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad (1.1)$$

This serves to define the electric field intensity $\vec{E}$ and the magnetic flux density $\vec{B}$ in terms of the charge q, the velocity of the charge relative to the observer $\vec{v}$, and the total force experienced by the charge $\vec{F}$. Consider a volume element ΔV which contains charges. If a charge density ρ, is defined as the limit of the ratio of the charge contained in ΔV to ΔV as $\Delta V \to 0$, and if a force density $\vec{f}$ is defined as the limit of $\vec{F}/\Delta V$ as $\Delta V \to 0$, then (1.1) becomes

$$\vec{f} = \rho_e \vec{E} + \rho_e \vec{v} \times \vec{B} \qquad (1.2)$$

The quantity $\rho_e \vec{v}$ represents a charge density in motion which is a current density $\vec{J}$. Throughout these notes it is implicitly assumed that there is no relative motion of components: therefore, convection currents which result from the motion of conductors are neglected. Equation (1.2) can therefore be written as

$$\vec{f} = \rho_e \vec{E} + \vec{J} \times \vec{B} \qquad (1.3)$$

The first term of this equation represents a force on static charges whereas the second is a force on moving charges or currents. In problems associated with magnet design, the

interactions between currents and magnetic fields are of primary interest. Forces due to the presence of free charge densities are usually negligible by comparison and (1.3) reduces to

$$\vec{f} = \vec{J} \times \vec{B} \qquad (1.4)$$

For any particular case, the validity of this approximation can be checked by evaluating $\rho_e \vec{E}$ and comparing it with the result of (1.4).

If it is postulated that charge must be conserved, the concepts of charge density and current density can be combined mathematically to represent a "law of conservation of charge" as follows

$$\oiint \vec{J} \cdot \vec{n} da = -\frac{d}{dt} \iiint \rho_e dV \qquad (1.5)$$

In (1.5) $\vec{n} da$ is an incremental area element of a closed surface and $\vec{n}$ is a normal, outwardly directed unit vector at the element. The incremental volume element within the closed surface is dV. Equation (1.5) therefore states that the net flow of charge out of the closed surface is equal to the rate of decrease of total charge within the enclosed volume. In the structural and electrical problems associated with magnet design, free space charge densities can usually be neglected and thus (1.5) reduces to

$$\oiint \vec{J} \cdot \vec{n} da = 0 \qquad (1.6)$$

The equivalent differential form of (1.6) is

$$\nabla \cdot \vec{J} = 0 \qquad (1.7)$$

Equations (1.6) and (1.7) normally enter magnet design problems implicitly rather than explicitly. Simply interpreted, both equations require that lines of current must form closed loops to be physically realizable.

The other conditions for physical realizability of the electric and magnetic fields were formulated by Maxwell as follows

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{n} da \tag{1.8}$$

$$\oint \vec{H} \cdot d\vec{l} - \frac{d}{dt} \iint \varepsilon_o \vec{E} \cdot \vec{n} da = \iint \vec{J} \cdot \vec{n} da \tag{1.9}$$

The first integral in each equation is taken around a closed contour having an incremental length $d\vec{l}$. The area integrals are taken over a simply connected surface bounded by a contour. The quantity $\vec{H}$ is the magnetic field intensity which in free space (that is, when magnetizable material is not present) is related to the magnetic flux density through $\vec{B} = \mu_o \vec{H}$.

The magnitude of the constant μ_o is dependent on the system of units employed. This quantity is the permeability of free space or vacuum, and has the value of $4\pi \times 10^{-7}$ H/m in the SI system. The constant ε_o is permittivity of free space which is a derived quantity. In SI units ε_o has a value of 8.854×10^{-12} F/m. The differential form of these equations is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.10}$$

$$\nabla \times \vec{H} - \varepsilon_o \frac{\partial \vec{E}}{\partial t} = \vec{J} \tag{1.11}$$

The second term in (1.9) and its counterpart in (1.11) represent what is called the displacement current. These terms are usually negligible in magnet design problems, as discussed in Section 1.2 Equations (1.9) and (1.11) therefore reduce to

$$\oint \vec{H} \cdot d\vec{l} = \iint \vec{J} \cdot \vec{n} da \tag{1.12}$$

$$\nabla \times \vec{H} = \vec{J} \tag{1.13}$$

If the divergence of (1.10) is taken, the result is

$$\nabla \cdot \left(\nabla \times \vec{E} \right) = -\nabla \cdot \left(\frac{\partial \vec{B}}{\partial t} \right) \tag{1.14}$$

However, since the divergence of the curl of any vector field is zero, and since the divergence and time operators are commutative, (1.14) becomes

$$\frac{\partial}{\partial t} \left(\nabla \cdot \vec{B} \right) = 0 \tag{1.15}$$

Equation (1.15) can be integrated with respect to time to yield

$$\nabla \cdot \vec{B} = \text{constant} \tag{1.16}$$

Therefore, the divergence of $\vec{B}$ is a quantity independent of time. Experimental evidence shows that this constant is zero. The last of the equations for physical realizability is therefore

$$\nabla \cdot \vec{B} = 0 \tag{1.17}$$

Note that this is a direct consequence of (1.10). In integral from, (1.17) can be expressed as

$$\oiint \vec{B} \cdot \vec{n} da = 0 \tag{1.18}$$

A simple interpretation of (1.17) and (1.18) is that magnetic flux lines must form closed loops to be physically realizable (no magnetic monopoles!).

Either the integral equations 91.6), (1.8), (1.12), and (1.18), or the differential equations (1.7), (1.10), (1.13), and (1.17) form the set of governing equations of interest in magnet design. The reduction of these equations or of the more general equations with displacement current terms to magnetostatics can be done directly by setting all terms involving a time derivative to zero. The conditions under which the displacement current terms can be neglected even though the situation involves variations in time is discussed in the following section.

1.2 Reduction to Magnet Systems and Magnetostatics

In order to illustrate conditions under which the displacement current term can be neglected, consider a region of space where

$$\vec{B} = \mu_o \vec{H} \tag{1.19}$$

$$\vec{J} = \sigma \vec{E} \tag{1.20}$$

That is, the magnetic flux density is related to $\vec{H}$ through the constant μ_o and the current density $\vec{J}$ is related to the electric field intensity $\vec{E}$ through the constant σ, which is the electrical conductivity.

If the curl of (1.11) is taken and (1.19) and (1.20) are used, then

$$\nabla \times \left(\nabla \times \vec{B} \right) = \mu_o \left(\nabla \times \vec{J} \right) + \mu_o \varepsilon_o \frac{\partial}{\partial t} \left(\nabla \times \vec{E} \right) \tag{1.21}$$

This can be rewritten using (1.10) and (1.20) as follows

$$\nabla \times \left(\nabla \times \vec{B} \right) = -\mu_o \sigma \frac{\partial \vec{B}}{\partial t} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} \tag{1.22}$$

This can be reduced using a vector identity[†] and (1.17), as follows

$$\nabla^2 \vec{B} - -\mu_o \sigma \frac{\partial \vec{B}}{\partial t} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \tag{1.23}$$

This is a wave equation which governs the behavior of the magnetic flux density $\vec{B}$ in materials with homogeneous isotropic conductivity and the permeability and permittivity of free space. Similar equations govern $\vec{E}$ and $\vec{J}$. Equation (1.23) forms the basis for\ study in areas such as waveguides and transmission lines. For the structural and electrical problems encountered in magnet design, the last term in (1.23) can be neglected which

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[†] $\nabla \times \nabla \times \vec{B} = \nabla \left(\nabla \cdot \vec{B} \right) - \nabla^2 \vec{B}$

reduces (1.23) to a diffusion equation. If the configuration being analyzed is also time invariant or varying slowly with time, then the second term can also be dropped which reduces the equation to the vector form of Laplace's equation. It is important to retain the distinction that it is the vector form.

The physical conditions under which some of the terms of (1.23) be neglected can be illuminated by casting the equation in dimensionless form. This can be accomplished by defining the following dimensionless variables.

$\hat{B} = \vec{B}/B_o$

$\hat{x}, \hat{y}, \hat{z} = x/l, y/l, z/l$

$\hat{\nabla} = l\nabla$

$\hat{t} = \omega t$

$B_o = $ characteristic magnetic flux density

$l = $ characteristic length

$\omega = $ characteristic frequency

If the above are substituted in 91.23) and terms rearranged, the result is that

$$\hat{\nabla}^2 \hat{B} - \mu_o \sigma \omega l^2 \frac{\partial \hat{B}}{\partial \hat{t}} - \mu_o \varepsilon_o \omega^2 l^2 \frac{\partial^2 \hat{B}}{\partial \hat{t}^2} = 0 \qquad (1.24)$$

The functional form for the solution of the dimensionless magnetic flux density is therefore

$$\hat{B} = \hat{B}(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \mu_o \varepsilon_o \omega^2 l^2) \qquad (1.25)$$

The solution is thus dependent on the dimensionless space and time variables as well as two additional dimensionless parameters. These two parameters describe the relative strength or importance of the second and third terms in comparison with the first term in the governing equation.

For example, consider the parameter associated with the third term of (1.24). This parameter can be rewritten using the fact that $\mu_o \varepsilon_o = 1/c^2$ where c is the velocity of light which is equal to the speed of an electromagnetic wave in free space.[4]

$$\mu_o \varepsilon_o \omega^2 l^2 = \frac{\omega^2 l^2}{c^2} \tag{1.26}$$

If l is a characteristic length of the component or material under consideration then l/c is the time required for the propagation of an electromagnetic disturbance or wave across this length. If the characteristic frequency ω with which the field is changing at a point is low, then the time $t = 2\pi/\omega$ associated with this change is long compared with the time required for propagation of the disturbance across the device to another point. If the frequency is low enough then the disturbance is, in effect, felt everywhere in the device at the same time, and the wave character of the problem can be neglected. That is, the third term in (1.23) can be ignored if

$$\frac{\omega^2 l^2}{c^2} << 1 \tag{1.27}$$

This is usually the case in magnet design. The third term in (1-24) is therefore neglected in the remainder of this section. In addition, the neglect of this term is an implicit assumption which is made for the analyses throughout these notes. The second terms of (1.9) and (1.11) are also neglected since they are the source of the wave character of the equation under discussion. Equation (1.24) therefore becomes

$$\hat{\nabla}^2 \hat{B} - \mu_o \sigma \omega l^2 \frac{\partial \hat{B}}{\partial \hat{t}} = 0 \tag{1.28}$$

This is a diffusion equation in which the strength of the diffusion term is determined by the magnitude of the dimensionless parameter $\mu_o \sigma \omega l^2$. This parameter is frequently called the magnetic Reynolds number. If the conditions of a particular problem are such that

$$\mu_o \sigma \omega l^2 << 1 \tag{1.29}$$

then the fields are essentially static or steady state in nature and (1.28) reduces to

$$\hat{\nabla}^2 \hat{B} = 0 \tag{1.30}$$

This is the form of the governing equation for magnetostatics. The concept of the magnetic Reynolds number is a useful tool which is considered in depth in a later section.

1.3 Boundary Conditions

The differential equations given in Sections 1.1 and 1.2 together with the constituent relations presented in Section 1.4 govern the relationship between the field variables in any region of space. If several regions having different properties are involved, boundary conditions are required to determine how the fields cross the surface which separates one region from another. These boundary conditions can be derived using the integral form of the equations given earlier. Since the primary interest is in magnet design, only those constituent relations and boundary conditions which are necessary for use with (1.7), (1.10), (1.13), and (1.17) are considered.

Two boundary conditions on the magnetic field must be considered, one to specify the relationship between the components of field normal to a boundary, and the other to specify the relationship between components tangent to a boundary. These can be found from (1.18) and (1.12) respectively.

First, (1.18) is applied to a small, closed cylindrical surface placed such that its faces are in two regions parallel to the boundary between regions, as shown in Figure 1.3-1. The dimensions of the cylinder are reduced about a point P located on the boundary which is within the cylindrical surface. The result is a condition which requires that the component of $\vec{B}$ normal to the boundary be continuous. This can be expressed mathematically as

$$\vec{n} \cdot \left(\vec{B}_2 - \vec{B}_1 \right) = 0 \qquad (1.31)$$

where $\vec{B}_1$ and $\vec{B}_2$ are the magnetic flux densities at P in Regions 1 and 2. respectively, and $\vec{n}$ is a unit vector at P normal to the boundary and directed from Region 1 to Region 2.

The second boundary condition can be found by applying (1.12) to a contour which surrounds a small plane perpendicular to the boundary between Regions 1 and 2. as shown in Figure 1.3-2. If $\vec{n}$ is again a unit vector at P which is normal to the boundary and directed from Region 1 to Region 2, then shrinking the dimensions about P results in

$$\vec{n} \times \left(\vec{H}_2 - \vec{H}_1 \right) = \vec{K} \qquad (1.32)$$

where $\vec{K}$ is a surface current density or current sheet which flows in the boundary. This is often absent in practical problems but is useful in certain idealized models. The boundary condition of (1.32) requires that the component of $\vec{H}$ tangential to the surface be discontinuous at the boundary if a current sheet exists, and that the discontinuity in $\vec{H}$ be equal in magnitude to the surface current density and at right angles to it.

1.4 Constituent Relations

In a typical magnetic field system, the conduction process accounts for the free current density in materials. The most common constituent relationship is Ohm's law,

$$\vec{J} = \sigma \vec{E} \qquad (1.33)$$

where σ is the electrical conductivity. The conductivity is typically assumed to be constant within a region, which requires that the conductivity in that region be homogeneous and isotropic. Throughout these notes it is implicitly assumed that there is no relative motion of the magnet components being analyzed.

Equation (1.33) is used in Section 1.2 together with the following constituent relationship which relates the magnetic flux density tot the magnetic field intensity in free space or in materials having the permeability of free space.

$$\vec{B} = \mu_o \vec{H} \qquad (1.34)$$

The simplest form of constituent relation for a magnetic material is to assume that the material is homogeneous and isotropic and that the field vectors are related by the permeability μ. which is constant, as follows

$$\vec{B} = \mu \vec{H} \qquad (1.35)$$

Often magnetic materials are used in which $\vec{B}$ is not directly proportional $\vec{H}$. For these the constitutive relationship takes the form

$$\vec{B} = \mu(H)\vec{H} \tag{1.36}$$

where $H = |\vec{H}|$. Some materials exhibit hysteresis which means that (1.36) is not single valued. In some applications this effect can be important; however, for many magnet designs a single-valued functional dependence is usually adequate. Data from which (1.36) can be derived is given in one of several forms. For example, the permeability may be plotted as a function of $\vec{H}$, that is, μ versus $\vec{H}$. Frequently, either $\vec{B}$ or the quantity $\left(\vec{B} - \mu_o\vec{H}\right)$ will be plotted versus $\vec{H}$.

The concept of magnetization arises in electromagnetic theory when the magnetic field is considered as being generated by two sources, one associated with an applied current density $\vec{J}$ which can be controlled directly, and the other associated with magnetizable material. The magnetizable material can be considered to consist of a source of current density $\vec{J}_m$ which is not controllable in that these magnetic currents cannot be circulated through an external circuit. With these two sources, (1.13) becomes

$$\nabla \times \left(\frac{\vec{B}}{\mu_o}\right) = \vec{J} + \vec{J}_m \tag{1.37}$$

If a magnetization density $\vec{M}$ is defined such that

$$\nabla \times \vec{M} = \vec{J}_m \tag{1.38}$$

then (1.37) can be rewritten as

$$\nabla \times \left(\frac{\vec{B}}{\mu_o} - \vec{M}\right) = \vec{J} \tag{1.39}$$

Comparison of this equation with (1.13) indicates that the magnetic field intensity $\vec{H}$ can be written in terms of the magnetization density as

$$\vec{H} = \left(\frac{\vec{B}}{\mu_o} - \vec{M} \right) \tag{1.40}$$

Thus a plot of the quantity $\left(\vec{B} - \mu_o \vec{H} \right)$ versus $\vec{H}$ is the same as a plot of $\mu_o \vec{M}$ versus $\vec{H}$.

This is a convenient formulation because $\vec{M}$ is frequently saturated when magnetic materials are used in large, high field magnets.

A magnetic susceptibility χ can also be defined such that

$$\vec{M} = \chi \vec{H} \tag{1.41}$$

The magnetic susceptibility is related to the permeability through

$$\mu = \mu_o \left(1 + \chi \right) \tag{1.42}$$

1.5 Potential Functions

Potential functions can be used to reduce the number of variables or otherwise simplify the process of finding a solution to the governing equations. The requirement of (1.17) that the divergence of $\vec{B}$ be zero naturally leads to the definition of a vector potential $\vec{A}$ of the form

$$\nabla \times \vec{A} = \vec{B} \tag{1.43}$$

This formulation automatically satisfies (1.17) since the divergence of the curl of any vector field is zero. For this analysis it is assumed that the constituent relations are given by (1.33) and (1.36). These require that (1) any electrically conducting material be homogeneous and isotropic with a constant electrical conductivity and (2) any magnetically permeable material be homogeneous and isotropic but not necessarily linear, since the permeability can be a function of $\vec{H}$. It is assumed that this function is

single valued, which requires that hysteresis effects be negligible. Equation (1.43) alone is not sufficient to define $\vec{A}$: however, if

$$\nabla \cdot \vec{A} = 0 \tag{1.44}$$

is imposed a constraint, $\vec{A}$ is fully defined.

Substitution of (1.43) into (1.10) leads to

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \tag{1.45}$$

and to the definition of a scalar potential function ϕ. This scalar potential automatically satisfies (1.45) since the curl of the gradient of a sclar function is zero. Therefore

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} = 0 \tag{1.46}$$

Equations (1.46) and (1.33) can be used with (1.7) to yield

$$\nabla \cdot \sigma \left(\nabla \phi + \frac{\partial \vec{A}}{\partial t} \right) = 0 \tag{1.47}$$

In addition, substitution of (1.33), (1.360, and (1.46) into (1.13) yields

$$\left(\nabla \times \frac{1}{\mu} \nabla \times \vec{A} \right) + \sigma \left(\nabla \phi + \frac{\partial \vec{A}}{\partial t} \right) = 0 \tag{1.48}$$

Equations (1.47) and (1.48) are the two governing equations in terms of the two unknown potential functions $\vec{A}$ and ϕ which can be used in place of the field equations for $\vec{E}$ and $\vec{B}$.

Equations (1.47) and (1.48) are the governing equations which should be used in the following manner for regions in which there are no current sources. If, as is often the case, the current density distribution is known in a region and is the driver in the particular configuration then the governing equation for that region becomes

$$\nabla \times \frac{1}{\mu} \nabla \times \vec{A} = \vec{J}_c \tag{1.49}$$

subject to (1.44). Note also that $\vec{J}_c$ must satisfy $\nabla \cdot \vec{J}_c = 0$ to be physically realizable. Equation (1.49) drives the solution in the other regions, which are governed by (1.47) and (1.48).

If the problem is two dimensional with a driving current density directed in the third dimension, the vector potential reduces to a single component in the direction of the driving current density. From (1.49) the governing equation in the current-carrying region becomes

$$\left(\nabla \cdot \frac{1}{\mu}\nabla\right)\vec{A} = -\vec{J}_c \tag{1.50}$$

From (1.48) the governing equation in the other regions is

$$\left(\nabla \cdot \frac{1}{\mu}\nabla\right)\vec{A} = \sigma\frac{\partial \vec{A}}{\partial t} \tag{1.51}$$

The advantage in this situation is that the vector potential $\vec{A}$ consists of a single component since $\vec{j}_c$ has only one component. If the two-dimensional problem is further simplified to consist of a steady-state situation, (1.50) remains unchanged but the right side of (1.51) becomes zero.

In a region of constant permeability μ, (1.13) and (1.43) require that

$$\nabla^2 \vec{A} = -\mu\vec{J}$$

In rectangular coordinates the components of this vector equation reduce to a scalar Laplacian, as follows:

$$\nabla^2 \vec{A}_x = -\mu\vec{J}_x$$

$$\nabla^2 \vec{A}_y = -\mu\vec{J}_y$$

$$\nabla^2 \vec{A}_z = -\mu\vec{J}_z \tag{1.53}$$

Note that each of these represents components of the vector equation (1.52) and that the vector form is equivalent to the scalar form of the differential equation only in

rectangular coordinates. In other coordinate systems, (1.52) does not reduce to the simple form of (1.53).

Each of the vector components must satisfy Poisson's equation which has a solution of the form[3]

$$A_i = \frac{\mu}{4\pi} \iiint \frac{\vec{J}_i dV}{r_{qp}}$$ (1.54)

where $i = x, y,$ or z and $r_{qp} =$ distance from point p where A_i is measured, to point q where $\vec{J}$ is measured.

This can then be written in vector form as

$$\vec{A} = \frac{\mu}{4\pi} \iiint \frac{\vec{J} dV}{r_{qp}}$$ (1.55)

where the integral is taken over the volume of the entire region. Equation (1.55) leads to a physical interpretation of $\vec{A}$. Consider a closed circuit of small-cross-section wire which carries a current density $\vec{J}$. Outside the wire there is no contribution to the volume integral of (1.55) because $\vec{J} = 0$. Inside the wire, $\vec{J}dV = Id\vec{s}$ where I is the current and $d\vec{s}$ is the length of the volume element in the direction of $\vec{J}$.
Equation (1.55) then becomes

$$\vec{A} = \frac{\mu}{4\pi} \oint \frac{Id\vec{s}}{r_{qp}}$$ (1.56)

Therefore, the vector contribution to $\vec{A}$ at a point by a current-carrying element in a closed circuit is parallel to that element.

Equations (1.43) and (1.55) can be manipulated to show that[4]

$$\vec{B} = \frac{\mu}{4\pi} \iiint \frac{\left(\vec{J} \times \vec{i}_{qp}\right)dV}{r_{qp}^2} \tag{1.57}$$

where $\vec{i}_{qp}$ is a unit vector directed from q to p. This equation is commonly called the Biot-Savart law.

In magnetostatic problems, conditions are steady in time. In current-free regions (1.13) becomes

$$\nabla \times \vec{H} = 0 \tag{1.58}$$

which is automatically satisfied by a scalar potential function defined by

$$\vec{H} = -\nabla \phi \tag{1.59}$$

Equations (1.59) and (1.17) together with (1.35) imply that the field can be found by solving

$$\nabla^2 \psi = 0 \tag{1.60}$$

where ψ is the magnetostatic potential.

1.6 Inductance and the Vector Potential

The mutual inductance between two circuits b and c can be thought of as the flux linked by c per unit current in b with zero current in c. Equation (1.43) defines the vector potential as

$$\nabla \times \vec{A} = \vec{B} \tag{1.61}$$

the integral form of this equation is

$$\oint_l \vec{A} \cdot d\vec{l} = \iint \vec{B} \cdot \vec{n} \, da \tag{1.62}$$

where l is a closed contour and the right-hand side is the flux through a simply-connected area bounded by l. If the contour coincides with circuit c which has zero current and if the field $\vec{B}$ and thus $\vec{A}$ are generated by a current I_b in circuit b, then the right side of (1.62) is the total flux generated by b and linked by c, as follows

$$\iint \vec{B} \cdot \bar{n} da = \Phi_{bc} \tag{1.63}$$

However, in linear systems

$$\Phi_{bc} = M_{bc} I_b \tag{1.64}$$

where M_{bc} is the mutual inductance between circuits b and c. Therefore

$$M_{bc} = \frac{1}{I_b} \oint_c \vec{A}_b \cdot d\vec{c} \tag{1.65}$$

The subscript is added to $\vec{A}$ in (1.65) to indicate that the vector potential is generated by the current I_b in circuit b although it is measured and integrated around circuit c which has zero current. This relationship can be used to obtain accurate inductance calculations through numerical integration.

1.7 Electromagnetic Forces

The force of electromagnetic origin is mentioned in Section 1.1 in connection with the definitions of $\vec{E}$ and of $\vec{B}$. In a magnet system, the electromagnetic loads are calculated by starting with (1.4) which gives the local force density $\vec{f}$ on an element which carries a current density $\vec{J}$ while immersed in a magnetic field $\vec{B}$. If the element and fields are such that $\vec{J}$ and $\vec{B}$ are uniform over the volume of the element as sketched in Figure 1.7.1, then the net force on the element is

$$\vec{F} = \left(\vec{J} \times \vec{B}\right) \text{dh dw dl} \tag{1.66}$$

If this incremental element is part of a larger volume of material carrying current in the magnetic field then the net force on any current-carrying volume or section can be found by integration as follows

$$\vec{F} = \iiint \vec{J} \times \vec{B} \, \text{dh dw dl} \qquad (1.67)$$

This equation can be used to find the net force on a current-carrying body by performing a volume integration. It can be useful to find the net force by performing an integral over a closed surface which surrounds the body. This involves the use of the Maxwell Stress Tensor, which is given by[4]

$$T_{mn} = \mu H_n H_m - \frac{\mu}{2} \delta_{mn} H_k H_k \qquad (1.68)$$

Where

$$\delta_{mn} = 1 \text{ when } m = n$$

$$\delta_{mn} = 0 \text{ when } m \neq n$$

and subscripts denote coordinate directions. Repeated subscripts such as k imply a summation. The mth component of the local force density is given by

$$f_m = \frac{\partial T_{mn}}{\partial x_n} \qquad (1.69)$$

and the net force on a body can be found from

$$F_m = \oiint T_{mn} n_n \, da \qquad (1.70)$$

where n_n, is the nth component of the outward-directed unit vector n normal to the closed surface surrounding the body.

The use of (1.67) requires that the fields be known throughout the volume of integration whereas (1.70) requires that they be known only on the closed surface. Furthermore, the closed surface can be considerably larger than the body on which the net force is desired.

This concept is particularly useful when the geometry of the problem is such that some of the components of the tensor or of the integrand in (1.70) are zero.

It can be shown that the form of (1.68) through (1.70) are the same whether the fields are generated by free current densities $\vec{J}$, by magnetized materials, or by a combination of these sources provided that the total field is used. The permeability μ must be isotropic, but can be a function of position. Furthermore, the permeability cannot be a function of the material density although this magnetostriction effect can be included in (1.68) through an additional term. These constraints are usually unimportant in fusion magnet systems.

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