Fall Term 2003
Plasma Transport Theory, 22.616
Problem Set #5
Prof. Molvig

Passed Out: Oct. 20, 2003 DUE: Nov. 4, 2003

1. **Fluctuation origin of U tensor:** Prove the identity used in class for the wavenumber, k-integral tensor,

$$\mathbf{I} = \int d^3k\delta\left(\mathbf{k}\cdot\left(\mathbf{v}-\mathbf{v}'\right)\right)\frac{\mathbf{kk}}{k^4\left|\epsilon\left(\mathbf{k},\mathbf{k}\cdot\mathbf{v}\right)\right|^2} = \pi\mathbf{U}\ln\Lambda$$

where, $\mathbf{U}$, is the tensor we encountered in the collision operator,

$$\mathbf{U} = \frac{1}{|\mathbf{v}-\mathbf{v}'|}\left(I - \frac{\left(\mathbf{v}-\mathbf{v}'\right)\left(\mathbf{v}-\mathbf{v}'\right)}{|\mathbf{v}-\mathbf{v}'|^2}\right)$$

and $\ln\Lambda$ is the Coulomb logarithm. In carrying out the integral, replace the dielectric function, $\epsilon\left(\mathbf{k},\mathbf{k}\cdot\mathbf{v}\right)$, by a *cutoff rule*, taking the residual integral from, $k_\perp^{\min}$ to, $k_\perp^{\max}$, where,

$$\frac{k_\perp^{\max}}{k_\perp^{\min}} = \frac{\lambda_{De}}{e^2/T} = n\lambda_{De}^3 = \Lambda$$

2. **Diffusion from plasma waves:** Now consider the case where a spectrum of plasma waves is excited somehow. Take a 1D case for simplicity. Assume that the frequency of all the waves is, $\omega = \omega_{pe}$, and that the wavenumber spectrum is flat in the narrow band from, $k\lambda_{De} = 2/10$, to, $k\lambda_{De} = 1/10$. This gives a wavenumber bandwidth of,

$$\Delta k = \frac{1}{10\lambda_{De}}$$

One can represent the spectrum in the following form,

$$\left\langle\left|\delta\phi_{k\omega}\right|^2\right\rangle = \frac{\phi_0^2}{\Delta k}\delta\left(\omega-\omega_{pe}\right)$$

Plot the electron particle diffusion coefficient resulting from these waves as a function of velocity, showing where it is non-zero. What would the effect of these waves be on the particle distribution?

Determine the magnitude of potential amplitude, $e\phi_0/T_e$, such that the resulting velocity diffusion is compareable to collisions. You should find,

$$\frac{e\phi_0}{T_e} \simeq \frac{1}{\sqrt{\Lambda}}$$

Estimate the implied fluctuation voltage, ϕ_0, in Volts, for a typical fusion plasma.

3. **Correlation Times:** Resolve the conundrum discussed in class wherein the resonance condition for diffusion, $\omega = \mathbf{k}\cdot\mathbf{v}$, implies zero frequency or infinite correlation time while the basic stochastic process principles say that an integrated stationary process must have zero correlation time to beget a diffusion process. Establish that the discreteness fluctuations we computed,

$$\left\langle \left|\delta\phi_{\mathbf{k}\omega}\right|^2 \right\rangle = \frac{2e^2}{\pi}\int d^3v' f\left(\mathbf{v}'\right)\delta\left(\omega - \mathbf{k}\cdot\mathbf{v}'\right)\frac{1}{k^4\left|\epsilon\left(\mathbf{k},\omega\right)\right|^2}$$

has a mean square frequency width, $\left\langle \omega^2 \right\rangle \simeq \Lambda\omega_{pe}^2$, where, $\Lambda = n\lambda_{De}^3$ is the plasma parameter. Do this by *estimating* the integral,

$$\left\langle \omega^2 \right\rangle \equiv \frac{\sum_{k,\omega}\omega^2\left\langle \left|\delta\phi_{\mathbf{k}\omega}\right|^2 \right\rangle}{\sum_{k,\omega}\left\langle \left|\delta\phi_{\mathbf{k}\omega}\right|^2 \right\rangle}$$

you may use a Maxwellian distribution for, $f\left(\mathbf{v}'\right)$, and take a simplified dielectric response function,

$$\epsilon\left(\mathbf{k},\omega\right) \simeq 1 + \frac{1}{k^2\lambda_{De}^2}$$

Don't worry about doing the integral exactly, just show that it comes out to this magnitude – note that this *correlation time is different – much smaller* – from proving my oft stated claim in class that the correlation time for collisions, τ_c, is of order, $\tau_c \sim 1/\omega_{pe}$. Give a physical explanation of the result.

4. **Turbulent Drift Wave Transport:** Consider low frequency drift waves, $\omega \ll \Omega_i$, potential fluctuations of the form,

$$\delta\phi\left(\mathbf{x},t\right) = \sum_{k_y,k_z,\omega}\delta\phi_{\mathbf{k}\omega}\exp\left(ik_y y + ik_z z - i\omega t\right)$$

in a slab geometry with, $\mathbf{B} = B\mathbf{e}_z$, density gradients in the x-direction only. Such fluctuations induce x-directed velocity fluctuations in the guiding center via the $\mathbf{E}\times\mathbf{B}$ drift,

$$\delta v_x\left(t\right) = \frac{c}{B}\delta E_y = \frac{c}{B}\sum_{k_y,k_z,\omega} -ik_y\delta\phi_{\mathbf{k}\omega}\exp\left(ik_y y\left(t\right) + ik_z z\left(t\right) - i\omega t\right)$$

By emulating the derivation of velocity space diffusion from class, show that this integrated process leads to *spatial* diffusion with coefficient,

$$D_{xx} = \pi\frac{c^2}{B^2}\sum_{k_y,k_z,\omega}k_y^2\left\langle \left|\delta\phi_{\mathbf{k}\omega}\right|^2 \right\rangle\delta\left(\omega - k_z v_z\right)$$

This can be expressed in terms of electron density fluctuations, δn_e, by using the *adiabatic response* expression (from thermal equilbrium for example),

$$\delta n_e = n\frac{e\delta\phi}{T}$$

yielding,

$$D_{xx} = \pi\left(\frac{cT}{eB}\right)^2\sum_{k_y,k_z,\omega}k_y^2\left\langle \left|\frac{\delta n_{\mathbf{k}\omega}}{n}\right|^2 \right\rangle\delta\left(\omega - k_z v_z\right)$$

Now make a quantitative estimate of the fluctuation level,

$$\left\langle \left| \frac{\delta n}{n} \right|^2 \right\rangle \equiv \sum_{k_y, k_z, \omega} \left\langle \left| \frac{\delta n_{\mathbf{k}\omega}}{n} \right|^2 \right\rangle$$

(as measured by a probe for example) required to give, $D_{xx} \simeq 1\ m^2/\sec$, when the drift wave spectrum has a frequency width, $\Delta\omega \simeq \omega_{*e} = k_y \rho_i v_{Ti}/L_n$, and characteristic wavenumbers, $k_y \rho_i \sim 1$. You may assume, $L_n/\rho_i \simeq 100$, and, $v_{Ti}/L_n \simeq 10^5\ \sec^{-1}$, and, $v_{Ti} \simeq 2{\times}10^7\ cm/\sec$. This is often referred to as the "$\delta n/n$" level (after the square root is taken), expresses as a percent.