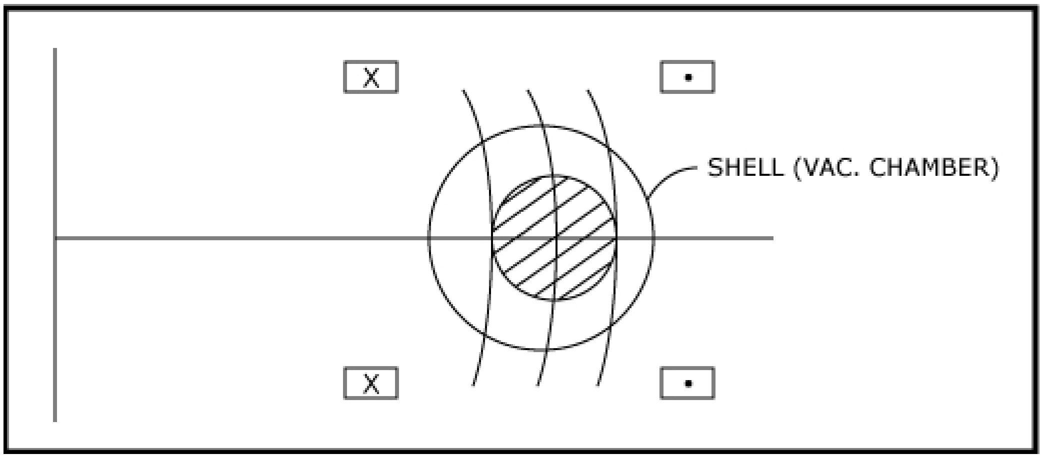
22.615, MHD Theory of Fusion Systems
Prof. Freidberg
Lecture 8: Effect of a Vertical Field on Tokamak Equilibrium

Toroidal Force Balance by Means of a Vertical Field

1. Let us review why the vertical field is important

2. 

3. For very short times, the vacuum chamber acts like a perfectly conducting shell: $t \sim 1$ msec.

4. On a longer time scale, the fields diffuse through the shell and a vertical field is required for equilibrium.

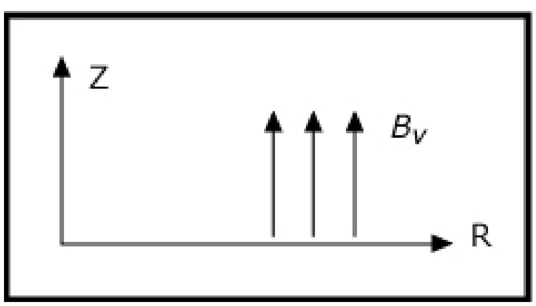
5. Analytic procedure: Assume an external vertical field slowly penetrates a highly conducting shell. The shell then becomes superconducting. We take the limit as the shell moves to infinity: $b \to \infty$.

6. The limit $b \to \infty$ is nontrivial. To begin, we leave the shell in place.

Influence of the Vertical Field

1. B_V causes a shift of the plasma surface with respect to R_0 , the center of the shell.

2. The applied vertical field is given by B= B_V e_Z



3. Assume B_V scales with the shift Δ. B_V is clearly a component of the poloidal field. Consider

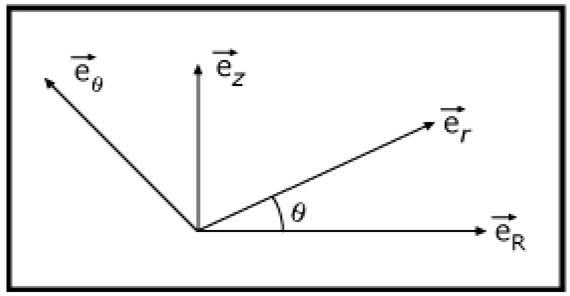
$$\frac{B_V}{B_\theta} \sim \in, \quad \frac{B_V}{B_0} \sim \in^2$$

4. Write B_V in terms of ψ_V, the vacuum vertical field flux function

   a. $\mathbf{B}_V = B_V \mathbf{e}_z$

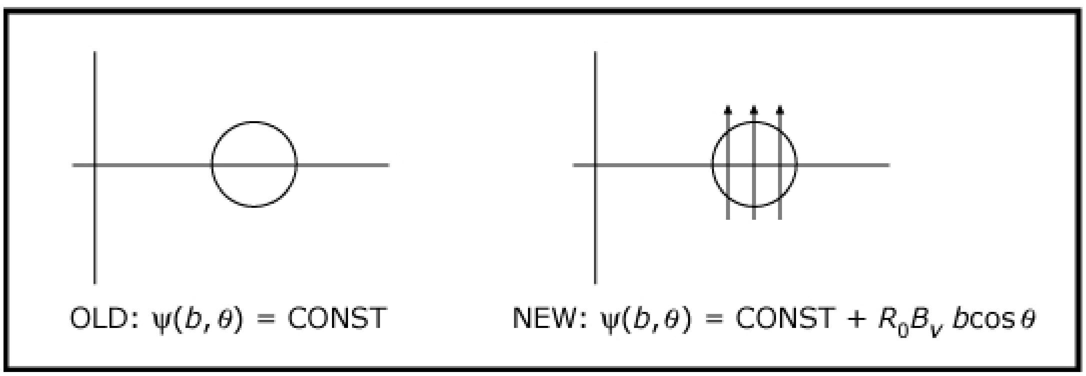
$$= B_V \left[\mathbf{e}_r \sin\theta + \mathbf{e}_\theta \cos\theta \right]$$

$$\equiv \frac{1}{R_0} \left[\frac{\partial \psi_V}{\partial r} \mathbf{e}_\theta - \frac{1}{r} \frac{\partial \psi_V}{\partial \theta} \mathbf{e}_r \right]$$



   b. $\psi_V = R_0 B_V r \cos\theta$

5. The new boundary condition including the vertical field is given by



Solve the Grad–Shafranov Equation Using the Tokamak Expansion to Account for *B_v*

1. Zero order: same as before: radial pressure balance

2. First order: same equation as before: $\nabla^2 \psi_1 = \ldots$ note $\cos\theta$ DEPENDENCE

   but with a new boundary condition: $\psi_1(b,\theta) = R_0 B_V b \cos\theta$

3. Let $\psi(r,\theta) = \psi_0(r) + \psi_1(r)\cos\theta \quad \psi_1(b) = R_0 B_V b$

4. The solution for ψ_1 is found as follows:

   a. $$\frac{d}{dr}\left(rB_\theta^2 \frac{d}{dr}\frac{\psi_1}{B_\theta}\right) = rB_\theta^2 - 2\mu_0 r^2 \frac{dp}{dr}$$

   b. $\psi_1 = \psi_1^{old} + \psi_1^{hom}$

      ψ_1^{old} satisfies the equation with B.C. $\psi_1^{old}(b) = 0$

      ψ_1^{hom} satisfies the homogeneous equation with B.C. $\psi_1^{hom}(b) = R_0 B_v b$

5. Homogeneous solution

   a. $$\left(\frac{\psi_1}{B_\theta}\right)' = \frac{c_1}{rB_\theta^2}$$

   b. $$\psi_1 = c_2 B_\theta + c_1 B_\theta \int_0^r \frac{dr}{rB_\theta^2}$$

   c. Choose $c_1 = 0$ for regularity

   d. $$\psi_1^{hom} = c_2 B_\theta(r) = \left[\frac{R_0 b B_v}{B_\theta(b)}\right] B_\theta(r)$$

6. The full solution can be written as

$$\psi_1(r) = -B_\theta \int_r^b \frac{dx}{xB_\theta^2}\int_0^x \left(yB_\theta^2 - 2y^2 \frac{dp}{dy}\right)dy + \frac{R_0 B_v b}{B_\theta(b)} B_\theta(r)$$

7. The new Shafranov shift is given by

   a. $$\Delta_a = -\frac{\psi_1(a)}{\psi_0'(a)} = -\frac{\psi_1^{old}(a)}{\psi_0'(a)} - \frac{\psi_1^{hom}(a)}{\psi_0'(a)}$$

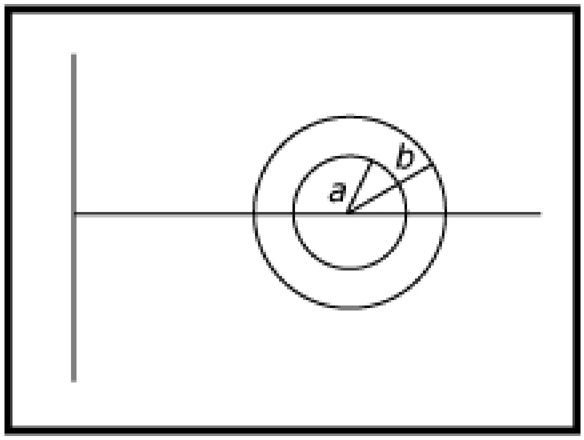
   b. $$\Delta_a = \Delta_{old} - \left[\frac{R_0 B_v b}{B_\theta(b)} B_\theta(a)\right]\frac{1}{R_0 B_\theta(a)}$$

$$= \Delta_{old} - \frac{bB_v}{B_\theta(b)}$$

8. Thus

$$\frac{\Delta_a}{b} = \frac{b}{2R_0}\left[\left(\beta_p + \frac{l_i}{2} - \frac{1}{2}\right)\left(1 - \frac{a^2}{b^2}\right) + \ln\frac{b}{a}\right] - \frac{B_V}{B_\theta(b)}$$

9. a. How much vertical field do we need to keep the plasma centered?



b. Set $\Delta_a = 0, B_\theta(b) = \left(\mu_0 I_p / 2\pi b\right)$

c. $B_v = \frac{\mu_0 I}{4\pi R_0}\left[\left(\beta_p + \frac{l_i}{2} - \frac{1}{2}\right)\left(1 - \frac{a^2}{b^2}\right) + \ln\frac{b}{a}\right]$

The Limit $b \to \infty$

1. How much field is required for $\Delta_a = 0$ if the shell *is not* present?

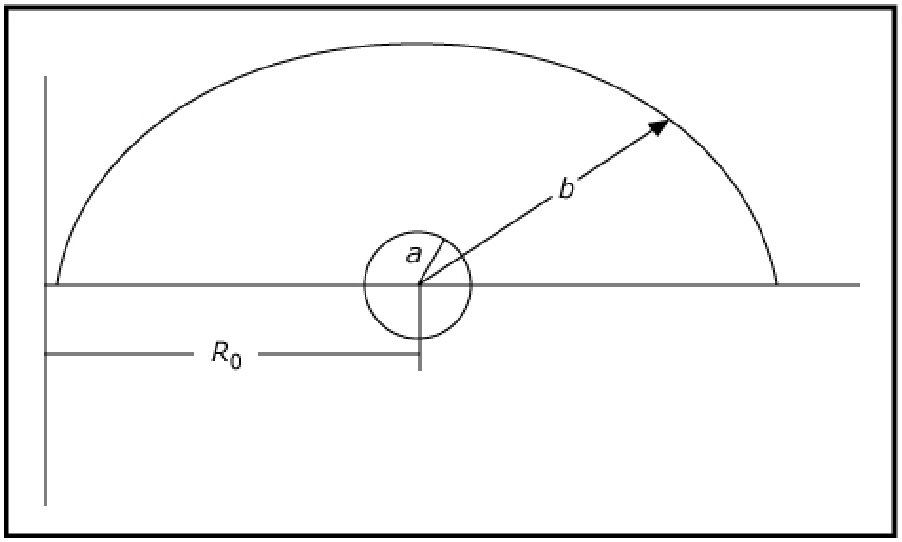
2. Imagine the shell receding further and further away so that $b/a \to \infty$

3. Take this limit in the expression for B_v

$$1 - \frac{a^2}{b^2} \to 1$$

$$\ln\frac{b}{a} \to ?$$

4. What is the difficulty?



a. Physically $b < R_0$. Also we assumed $b/R_0 \ll 1$

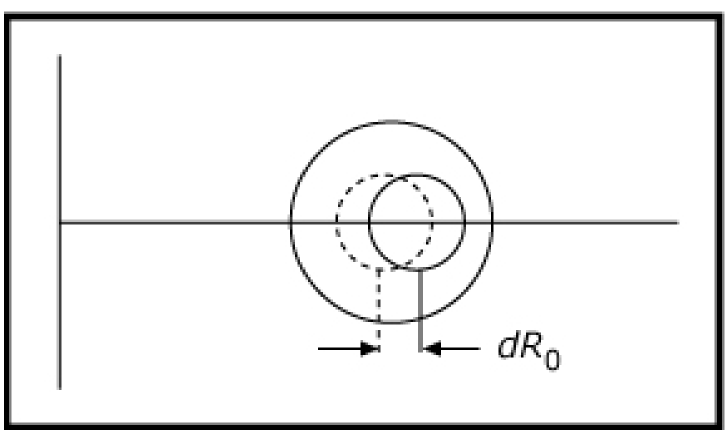
b. Here is a simple approximation: $\ln(b/a) \rightarrow \ln(R_0/a)$

Electrical Engineering Derivation of the ln b / a Limit

1. ln *b/a* represents the force due to the change in magnetic energy between the plasma and the wall as the plasma shifts outward by an amount $dR_0 = \Delta$

2. It is the analog of the l_i term except applied to the external field

3. External field changes:



4. The force $= -\nabla$ (potential energy) $= -\nabla$ (magnetic energy) as the plasma is displaced by an amount dR_0. Note that since the plasma is a perfect conductor, the flux linking the plasma remains fixed during the displacement

   a. $$F = -\frac{d}{dR_0}\left[\frac{1}{2\mu_0}\int B_\theta^2 dr\right] = -\frac{d}{dR_0}\left(\frac{1}{2}L_e I^2\right)$$ (L_e: external inductance)

   b. Constant flux implies $L_e I = \psi_e =$ const.

   c. $$F = -\frac{L_e^2 I^2}{2}\frac{d}{dR}\frac{1}{L_e} = \frac{I^2}{2}\frac{dL_e}{dR_0}$$

5. If the plasma is surrounded by a shell

   $$L_e = \mu_0 R_0 \ln b/a$$

   $$F = \frac{\mu_0 I^2}{2}\ln\frac{b}{a}$$

6. For a plasma without a shell (homework problem)

   $$L_e = \mu_0 R_0\left[\ln\frac{8R_0}{a} - 2\right]$$

   $$F = \frac{\mu_0 I^2}{a}\left[\ln\frac{8R_0}{a} - 1\right]$$

7. Therefore, the proper limit is

   $$\ln\frac{b}{a} \to \ln\frac{8R_0}{a} - 1 = \ln\frac{R_0}{a} + 1.08$$

8. Substitute into the B_v formula

   $$B_v = \frac{\mu_0 I}{4\pi R_0}\left[\beta_p + \frac{l_i}{2} - \frac{3}{2} + \ln\frac{8R_0}{a}\right]$$

9. This is a widely used formula in the design of circular tokamaks

Summary of Ohmically Heated Tokamak Equilibria

1. low β : $\beta_t \sim \epsilon^2, \beta_p \sim 1$

2. $q \sim 1$: required for stability

3. radial pressure balance: poloidal field (Z pinch)

4. toroidal force balance: vertical field B_v

5. toroidal field: needed only for stability to keep $q \sim 1$

6. Ordering

   $\beta_t \sim \epsilon^2$

   $\beta_p \sim 1$

   $q \sim 1$

   $B_\theta / B_0 \sim \epsilon$

   $\delta B_\phi / B_0 \sim \epsilon^2$

Intuitive Form of Toroidal Force Balance

1. Multiply the B_v equation by $2\pi R_0 I$

2. Then

$$2\pi R_0 I B_v = \frac{\mu_0 I^2}{2}\left[\beta_p + \frac{l_i}{2} - \frac{3}{2} + \ln\frac{8R_0}{a}\right]$$

3. $T_1 = 2\pi R_0 I B_v = B_v I L$ force due to the vertical field acting on the current I

4. $T_2 = \frac{\mu_0 I^2}{2}\left[\ln\frac{8R_0}{a} - 1\right] = \frac{I^2}{2}\frac{dL_e}{dR}$ force due to the change in the external field

5. $T_3 = \frac{\mu_0 I^2}{2}\frac{l_i}{2} = \frac{\mu_0 I^2}{4}\frac{L_i}{2\pi R_0}\frac{4\pi}{\mu_0} = \frac{I^2}{2}\frac{L_i}{R_0}$

   a. but

$$\frac{1}{2}L_i I^2 = \int \frac{B_\theta^2}{2\mu_0} d\mathrm{r} = \frac{2\pi^2 R_0}{\mu_0}\int B_\theta^2 r dr$$

$$L_i = \underbrace{\left[\frac{4\pi^2}{\mu_0}\int\left(\frac{B_\theta}{I}\right)^2 r\ dr\right]}_{\text{independent of } R_0} \times R_0$$

b. Thus $L_i \propto R_0$ and $\frac{L_i}{R_0} = \frac{dL_i}{dR_0}$ so that $\frac{\mu_0 I^2}{2}\frac{l_i}{2} = \frac{I^2}{2}\frac{dL_i}{dR_0}$

c. $T_3 = \frac{I^2}{2}\frac{dL_i}{dR_0}$

6. $T_4 = \frac{\mu_0 I^2}{2}\left[\beta_p - \frac{1}{2}\right] = \frac{\mu_0 I^2}{2}\left[\frac{16\pi^2}{\mu_0 I^2}\int pr\,dr - \frac{1}{2}\right]$

$$= 8\pi^2 \int pr\;dr - \frac{\mu_0 I^2}{4}$$

a. Recall the general pressure balance relation

$$2\pi\int pr\,dr = \frac{\mu_0 I^2}{8\pi} + 2\pi\int r\,\frac{B_0^2 - B_\phi^2}{2\mu_0}\,dr$$

let $B_\phi(r) = B_0 + B_2(\psi_0) \equiv B_0 + \delta B_\phi(r)$

$$\frac{\mu_0 I^2}{4} = 4\pi^2\int pr\,dr + 4\pi^2\int r\,\frac{B_0 \delta B_\phi}{\mu_0}\,dr$$

b. $T_4 = 8\pi^2\int pr\,dr - 4\pi^2\int pr\,dr - 4\pi^2\int r\,dr\,\frac{B_0\delta B_\phi}{\mu_0}$

$$= 4\pi^2\int\left(p - \frac{B_0\delta B_\phi}{\mu_0}\right)r\;dr$$

7. Summary

$$2\pi R_0 I B_v = \frac{I^2}{2}\frac{d}{dR_0}\left(L_e + L_i\right) + 4\pi^2\int\left(p - \frac{B_0\delta B_\phi}{\mu_0}\right)r\,dr$$

vertical field force | hoop force | tire tube force | $1/R$ force

8. Proof of the tire tube and $1/R$ force

a. F_{tt} :

$$F_{tt} = -\int e_R \cdot \nabla p\,dr$$

$$= -\int \left[\underbrace{\nabla \cdot \left(\mathbf{e}_R p \right)}_{\text{integrates to zero}} - p \overbrace{\nabla \cdot \mathbf{e}_R}^{\frac{1}{R}} \right] d\mathbf{r}$$

$$= \int \frac{p}{R} R r d\phi \; dr \; d\theta$$

$$= 4\pi^2 \int p r \, dr$$

b. $F_{1/R}$

$$\mathbf{J}_p \times \mathbf{B}_\phi \cdot \mathbf{e}_R = \frac{1}{R\mu_0} \left[\nabla R B_\phi \times \mathbf{e}_\phi \right] \times B_\phi \mathbf{e}_\phi \cdot \mathbf{e}_R = -\frac{B_\phi}{R\mu_0} \mathbf{e}_R \cdot \nabla R B_\phi$$

$$= -\frac{1}{R^2} \mathbf{e}_R \cdot \nabla \frac{R^2 B_\phi^2}{2\mu_0}$$

$$= -\frac{1}{R^2} \mathbf{e}_R \cdot \nabla \frac{F^2}{2\mu_0} = -\frac{1}{R^2} \mathbf{e}_R \cdot \nabla \frac{R_0^2}{2\mu_0} \left(B_0^2 + 2 B_0 \delta B_\phi \right)$$

$$= -\frac{R_0^2}{\mu_0 R^2} \mathbf{e}_R \cdot \nabla B_0 \delta B_\phi$$

c. $F_{1/R} = \int \left(\mathbf{J}_p \times \mathbf{B}_\phi \cdot \mathbf{e}_R \right) d\mathbf{r}$

$$= -\frac{R_0^2}{\mu_0} \int \frac{1}{R^2} \mathbf{e}_R \cdot \nabla B_0 \delta B_\phi d\mathbf{r}$$

$$= -\frac{R_0^2}{\mu_0} \int \left[\nabla \cdot \left(\frac{B_0 \delta B_\phi}{R^2} \mathbf{e}_R \right) - B_0 \delta B_\phi \overbrace{\nabla \cdot \frac{\mathbf{e}_R}{R^2}}^{-\frac{1}{R^3}} \right] d\mathbf{r}$$

$$= -\frac{R_0^2}{\mu_0} \int \frac{B_0 \delta B_\phi}{R^3} d\mathbf{r} \approx -\frac{R_0^2}{\mu_0} \int \frac{B_0 \delta B_\phi}{R^3} R r d\phi \; d\theta \, dr$$

$$\approx -4\pi^2 \int \frac{B_0 \delta B_\phi}{\mu_0} r \, dr$$