22.615, MHD Theory of Fusion Systems
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Lecture 7: The First Order Grad–Shafranov Equation

First Order Equation

The first order Grad–Shafranov equation is given by

$$\nabla^2 \psi_1 + \left[\mu_0 R_0^2 \frac{d^2 p}{d\psi_0^2} + R_0^2 \frac{d^2}{d\psi_0^2} B_0 B_2 \right] \psi_1 = -2\mu_0 R_0 \frac{dp}{d\psi_0} r \cos\theta + \frac{1}{R_0} \frac{d\psi_0}{dr} \cos\theta$$

Simplify

This equation is simplified as follows:

1. $B_\theta = \frac{1}{R_0} \frac{d\psi_0}{dr}, \quad \psi_0 = \psi_0(r)$

2. $R_0 \frac{dp}{d\psi_0} = \frac{R_0}{\psi_0'} \frac{dp}{dr} = \frac{1}{B_\theta} \frac{dp}{dr}$

3. $RHS = \left[B_\theta - \frac{2\mu_0}{B_\theta} r \frac{dp}{dr} \right] \cos\theta$

4. $[\ ] = R_0^2 \frac{1}{R_0 B_\theta} \frac{d}{dr} \left[\frac{1}{R_0 B_\theta} \frac{d}{dr} \left(\mu_0 p + B_0 B_2 \right) \right]$

$$= -\frac{1}{B_\theta} \frac{d}{dr} \left(\frac{1}{B_\theta} \frac{B_\theta}{r} \frac{d}{dr} r B_\theta \right) = -\frac{1}{B_\theta} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} r B_\theta \right)$$

5. $\nabla^2 \psi_1 - \frac{1}{B_\theta} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} r B_\theta \right) \psi_1 = \left[-\frac{2\mu_0 r}{B_\theta} \frac{dp}{dr} + B_\theta \right] \cos\theta$

Solve

This equation is solved as follows:

1. Note that all the forcing terms are proportional to $\cos\theta$

2. a. The boundary conditions for a *circle* of radius *b* are given by

$$\psi(b, \theta) = \text{const} = \psi_0(b) + \psi_1(b, \theta)$$

$$\psi_1(b, \theta) = 0$$

b. For an *ellipse* $r = b\left[1+\delta\cos 2\theta\right]$. Assume $\delta \sim \in \ll 1$.

$$\psi\left[b + b\delta\cos 2\theta, \theta\right] \approx \psi_0\left(b\right) + b\psi_0^{'}\delta\cos 2\theta + \psi_1\left(b,\theta\right) + \ldots$$

$\psi_1\left(b,\ \theta\right) = -b\psi_0^{'}\delta\cos 2\theta$ a second harmonic is required in the solution

3. Thus, for a circular boundary with $\cos\theta$ driving terms we can write

$\psi\left(r,\theta\right) = \psi_0\left(r\right) + \psi_1\left(r\right)\cos\theta$ explicit $\cos\theta$ dependence

$$\psi_1\left(b\right) = 0$$

4. Simplify the ψ_1 equation

$$\frac{1}{r}\left(r\psi_1^{'}\right)^{'} - \frac{\psi_1}{r^2} - \frac{1}{B_\theta}\left[B_\theta^{''} + \frac{B_\theta^{'}}{r} - \frac{B_\theta}{r^2}\right]\psi_1 = B_\theta - \frac{2\mu_0 r p^{'}}{B_\theta}$$

$$\frac{1}{r}\left(r\psi_1^{'}\right)^{'} - \frac{\left(rB_\theta^{'}\right)^{'}}{rB_\theta}\psi_1 = B_\theta - \frac{2\mu_0 r}{B_\theta}p^{'}$$

5. Note

$$\left[rB_\theta^2\left(\frac{\psi_1}{B_\theta}\right)^{'}\right]^{'} = \left(rB_\theta\psi_1^{'} - r\psi_1 B_\theta^{'}\right)^{'}$$

$$= B_\theta\left(r\psi_1^{'}\right)^{'} + rB_\theta^{'}\psi_1^{'} - \psi_1\left(rB_\theta^{'}\right)^{'} - r\psi_1^{'}B_\theta^{'}$$

6. $$\frac{d}{dr}\left(rB_\theta^2\frac{d}{dr}\frac{\psi_1}{B_\theta}\right) = rB_\theta^2 - 2\mu_0 r^2\frac{dp}{dr}$$

7. $$\frac{d}{dr}\frac{\psi_1}{B_\theta} = \frac{1}{rB_\theta^2}\int_0^r\left(yB_\theta^2 - 2\mu_0 y^2\frac{dp}{dy}\right)dy$$

regularity at r = 0

8. $$\psi_1 = -B_\theta\int_r^b\frac{dx}{xB_\theta^2}\int_0^x\left(yB_\theta^2 - 2\mu_0 y^2\frac{dp}{dy}\right)dy$$

9. This expression for ψ_1 represents the toroidal correction to the equilibrium solution.

Consequences of Toroidicity

1. The main consequence is an outward shift of the flux surfaces.

2. From ψ_1 it is straightforward to calculate $\Delta(r)$ the flux surface shift.

3. a. $\psi(r,\theta) = \psi_0(r) + \psi_1(r)\cos\theta = \text{const.}$

   b. Assume the flux surfaces are of the form $r = r_0 + r_1(r_0,\theta)$.

   c. Then $\psi_0(r_0) + \psi_0'(r_0)r_1 + \psi_1(r_0)\cos\theta ... = \text{const.}$

   d. Solve for r_1

   $$r_1(r,\theta) = -\frac{\psi_1(r)\cos\theta}{\psi_0'(r)} = \Delta(r)\cos\theta$$

   $$\Delta(r) \equiv -\frac{\psi_1(r)}{\psi_0'(r)}$$

4. The equation for the flux surfaces is given by $r = r_0 + \Delta(r_0)\cos\theta$, assuming $\Delta \ll r_0$

5. a. Note that $(x-\Delta)^2 + (y)^2 = r_0^2$ is the equation of a shifted circle, equivalent to the equation for r.

   b. Let

   $$x = r\cos\theta \quad y = r\sin\theta$$

   $$r^2 - 2r\Delta\cos\theta \approx r_0^2$$

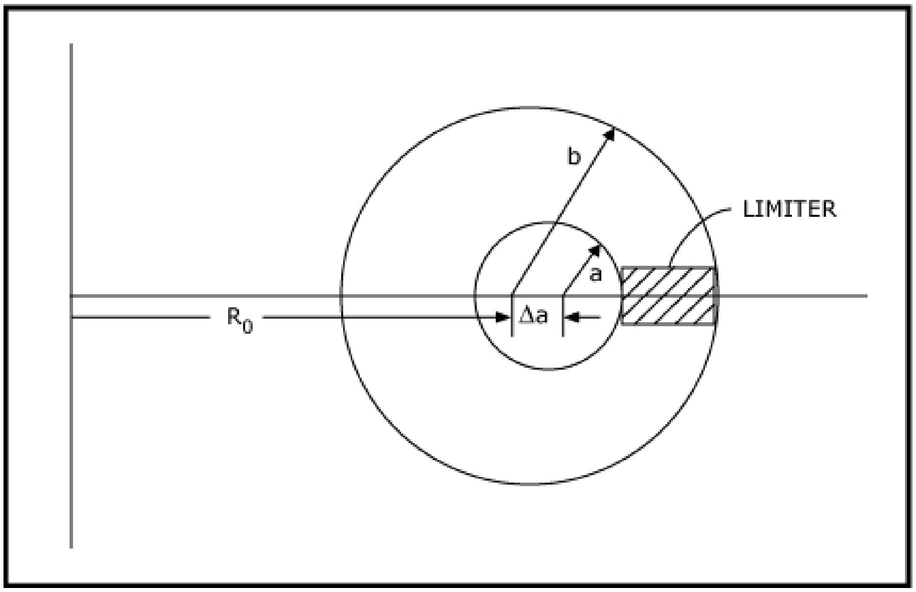
   $$r\left[1 - \frac{\Delta}{r}\cos\theta\right] \approx r_0$$

   $$r \approx r_0 + \Delta\cos\theta$$

6. The flux surfaces are shifted circles, with shift $\Delta = -\psi_1/\psi_0'$

The Shafranov Shift

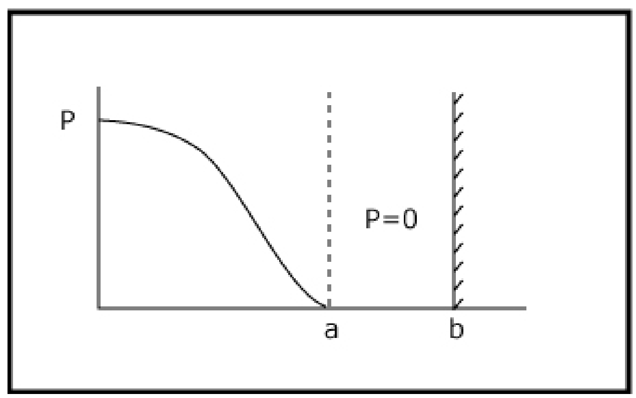
1. Calculate the Shafranov shift $\Delta_a \equiv \Delta(a)$ where a is the last surface to carry current, i.e., the edge of the plasma.



2. Simplify $\psi_1(a)$ given by

$$\psi_1(a) = -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x \left(yB_\theta^2 - 2\mu_0 y^2 \frac{dp}{dy}\right) dy$$

3. Consider the term T_1, noting that $a < x < b$



a. $$-\mu_0\int_0^x 2y^2 p' dy = -\mu_0\int_0^a 2y^2 p' dy$$

$$= -2\mu_0 y^2 p\Big|_0^a + 4\int_0^a \mu_0 py \; dy$$

$$= \frac{\mu_0^2 I^2}{4\pi^2}\beta_p$$

$$= a^2 B_\theta^2(a)\beta_p$$

b. $$T_1 = -a^2 B_\theta^3(a)\beta_p\int_a^b \frac{dx}{xB_\theta^2}$$

c. For $x > a$ $\;B_\theta = B_\theta(a)\dfrac{a}{x}$

d. Therefore

$$T_1 = -B_\theta(a)\beta_p \int_a^b x\,dx$$

$$= -\frac{B_\theta(a)\beta_p b^2}{2}\left(1-\frac{a^2}{b^2}\right)$$

4. Consider the term T_2

   a. Separate the integral into two parts

$$T_2 = -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x yB_\theta^2 dy$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left[\int_0^a yB_\theta^2 dy + \int_0^x yB_\theta^2 dy\right]$$

   b. T_{2b} : $\int_a^x yB_\theta^2 dy = B_\theta^2(a)\,a^2\int_a^x \frac{dy}{y} = a^2 B_\theta^2(a)\left[\ln x - \ln a\right]$

   c. Substitute

$$T_{2b} = -B_\theta(a)\,x\left[\ln x - \ln a\right]$$

$$= -B_\theta(a)\left[-\left(\frac{b^2 - a^2}{2}\right)\ln a + \frac{b^2}{2}\ln b - \frac{b^2}{4} - \frac{a^2}{2}\ln a + \frac{a^2}{4}\right]$$

$$= B_\theta(a)\frac{b^2}{4}\left[1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right]$$

   d. $T_{2a} = -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^a yB_\theta^2 dy$

   e. Introduce the normalized internal inductance per unir length l_i.

   f. This follows from

$$\frac{1}{2}L_i I^2 = \int_p \frac{B_\theta^2}{2\mu_0}dr$$

$$= (2\pi R_0)(2\pi)\frac{1}{2\mu_0}\int B_\theta^2 r\; dr$$

$$= \frac{2\pi^2 R_0}{\mu_0}\int_0^a B_\theta^2 r\,dr$$

g. Define $l_i \equiv \frac{L_i}{2\pi R_0} \Big/ \frac{\mu_0}{4\pi}$ the internal inductance per unit length normalized to $\mu_0/4\pi$

h. Then

$$\int_0^a B_\theta^2 r\,dr = \left(\frac{\mu_0}{2\pi^2 R_0}\right)\frac{1}{2}\left(2\pi R_0 l_i\right)\frac{\mu_0}{4\pi}\left[\frac{2\pi a B_\theta(a)}{\mu_0}\right]^2$$

$$= \frac{1}{2}a^2 B_\theta^2(a)\, l_i$$

i. l_i is a profile parameter related to the width of the J_ϕ profile

j. Then

$$T_{2a} = -B_\theta(a)\frac{l_i}{2}\int_a^b x\; dx$$

$$= -B_\theta(a)\frac{l_i}{4}b^2\left(1-\frac{a^2}{b^2}\right)$$

5. Combine these terms to evaluate Δ_a

a. $\Delta_a = -\frac{\psi_1(a)}{\psi_0'(a)} = -\frac{\psi_1(a)}{R_0 B_\theta(a)}$

b. Then

$$\Delta_a = -\frac{1}{R_0 B_\theta(a)}\left[-\frac{B_\theta(a)\beta_p b^2}{2}\left(1-\frac{a^2}{b^2}\right)+B_\theta(a)\frac{b^2}{4}\left(1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right)-B_\theta(a)\frac{b^2 l_i}{4}\left(1-\frac{a^2}{b^2}\right)\right]$$

c. The Shafranov shift is given by

$$\frac{\Delta_a}{b} = \frac{b}{2R_0}\left[\left(\beta_p+\frac{l_i}{2}-\frac{1}{2}\right)\left(1-\frac{a^2}{b^2}\right)+\ln\frac{b}{a}\right]$$

Properties of the Shafranov Shift

1. $\frac{\Delta_a}{b} \sim \frac{b}{R_0} \sim \epsilon \ll 1$. The shift is small, implying that our approximations are consistent.

2. $\frac{\Delta_a^{(1)}}{b} \propto \beta_p$. This is the outward shift due to the tire tube force and the $1/R$ force.

3. $$\frac{\Delta_a^{(2)}}{b} \propto \underbrace{\frac{l_i}{2}\left(1+\frac{a^2}{b^2}\right)}_{\text{internal field}} + \underbrace{\ln\frac{b}{a} - \frac{1}{2}\left(1-\frac{a^2}{b^2}\right)}_{\text{external field}}$$

internal field external field

hoop force

This is the shift due to the hoop force.