Lecture 3: MHD Equilibrium

Derivation of the MHD Equilibrium Equation

1. Focus on the fluid momentum equations.

2. Combine them in a clever way to obtain the basic equation of MHD equilibrium.

3. In the derivation we will have to make a few assumptions, but these are well satisfied for the general problems of interest – MHD equilibrium and stability, and transport.

4. Recall that the fluid momentum equations are given by

$$m_j n_j \frac{d\mathbf{u}_j}{dt} + \nabla \cdot \ddot{\mathbf{P}}_j - q_j n_j \left(\mathbf{E} + \mathbf{u}_j \times \mathbf{B} \right) = \sum_k^I \mathbf{R}_{jk} - m_j \mathbf{u}_j \mathbf{S}_{nj}$$

5. Sum these equations over all species, noting that

$$\sum_j \ddot{\mathbf{P}}_j \equiv \ddot{\mathbf{P}} \quad \text{total pressure tensor for all species}$$

$$\sum_j q_j n_j \mathbf{E} = \sigma \mathbf{E}$$

$$\sum_j q_j n_j \mathbf{u}_j \times \mathbf{B} = \mathbf{J} \times \mathbf{B}$$

$$\sum_{j,k}^I \mathbf{R}_{jk} = \sum_{j,k}^I \int m_j \mathbf{w} C_{jk} d\mathbf{w} = 0$$

6. Then

$$\sum_j m_j n_j \frac{du_j}{dt} = \mathbf{J} \times \mathbf{B} + \sigma \mathbf{E} - \nabla p - \nabla \cdot \overleftrightarrow{\Pi} - \sum_j m_j u_j S_{nj}$$

7. For all problems of interest we assume the plasma is sufficiently dense so that charge neutrality applies. Recall that this requires $\lambda_D \ll a, \omega \ll \omega_{pe}$, which are well satisfied for MHD and transport phenomena. Charge neutrality implies that

$$\sigma = \sum_j q_j n_j \approx 0$$

Thus, the term $\sigma \mathbf{E}$ is the negligibly small.

8. During the course we will be interested in two main classes of problems

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a. MHD equilibrium and stability – perturbation analysis about a static equilibrium.

b. Transport phenomena including diffusion, heat conduction, and flux penetration.

9. For these classes of problems, the fluid flow velocities are either zero or very small.

   a. For static MHD equlibria, $u_i \approx 0$ and all current is carried by electrons.

   b. This implies that the inertial terms and source terms are negligible. For example

   $$\sum_j m_j n_j \frac{d u_j}{dt} \sim m_e n_e u_e \cdot \nabla u_e \sim \frac{m_e n_e u_e^2}{a}$$

   Substitute $J \sim e n_e u_e$ and compare to the $J \times B$ term

   $$\frac{m_e n_e u_e^2}{aJB} \sim \frac{m_e J}{e^2 n_e aB} \sim \left(\frac{c^2}{a^2 \omega_{pe}^2}\right)\left(\frac{\mu_0 aJ}{B}\right) \lll 1$$

   c. A similar argument holds for transport phenomena, although we must now consider a nonzero ion fluid velocity associated with diffusion. This velocity is also very small. Typically

   $$n u = -D \nabla n \rightarrow u \sim D/a \quad \text{flow velocity}$$

   $$\frac{\partial}{\partial t} \sim \frac{\chi}{a^2} \quad \text{fastest transport time scale}$$

   $$\sum_j n_j m_j \frac{d u_j}{dt} \sim n_i m_i \frac{\partial u_i}{\partial t}$$

   Compare this term with ∇p

   $$\frac{m_i n_i}{\nabla p} \frac{\partial u_i}{\partial t} \sim \frac{n_i m_i}{p} \frac{D\chi}{a^2} \sim \frac{a^2}{v_{Ti}^2} \frac{D\chi}{a^4} \sim \frac{\tau_t^2}{\tau_p \tau_E} \lll 1$$

   d. The conclusion is that the inertia and source terms can be neglected. The momentum equation reduces to

   $$J \times B = \nabla p + \nabla \cdot \overset{\leftrightarrow}{\Pi}$$

10. The $\overset{\leftrightarrow}{\Pi}$ matrix is also small under the situations of interest. There are two types of terms which appear in $\overset{\leftrightarrow}{\Pi}$, viscosity and pressure anisotropy. Since

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viscosity is proportional to $\mu\nabla^2u$, this term is small for zero or slow flows. It can easily be shown that

$$\frac{\mu_e\nabla^2u_e}{J\times B} \sim \left(\frac{\tau\,Le}{a}\right)^2 \left(\Omega_e\tau_{ei}\right) \lll 1$$

$$\frac{\mu_i\nabla^2u_i}{\nabla p} \sim \frac{\tau_{ii}}{\tau_p} \lll 1$$

11. A similar calculation shows that the pressure anisotropy is very small, unless external sources are applied that deliberately drive an anisotropy. We shall not consider such situations. The time scale for any anisotropy to relax is typically τ_{ee} for electrons and τ_{ii} for ions. Thus, since

$$\frac{\tau_{ii}}{\tau_p} \lll 1$$

the plasma should be isotropic for the situations of interest.

12. The conclusion is that the MHD equilibrium equation

$$J\times B = \nabla p$$

is a very accurate representation of

a. The initial equilibrium state of a static plasma, which is to be tested for MHD instability.

b. The sequence of quasi-static states that a plasma evolves through as it slowly evolves on the transport time scale.

Goals of MHD Equilibrium Studies

1. Find configurations which confine and isolate hot plasmas from material walls.

2. Find configurations which have good stability properties at high β.

Why is β so important in fusion?

1. Consider the simple energy balance consisting of

a. Alpha power in a thermonuclear plasma $P_\alpha = \left(Q_\alpha/4\right)n^2\overline{\sigma v}$ watts/m^3.

b. Energy loss $P_l = \left(3nT_i/\tau_E\right)$ watts/m^3.

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2. For successful ignition we need $P_\alpha > P_l$:

$$n\tau_E > \frac{12 T_i}{Q_\alpha \overline{\sigma v}}$$

3. Define $\beta = 2\mu_0 n\left(T_e + T_i\right)/B^2 = 4\mu_0 n T_i/B^2$ for $T_e = T_i$. Here $\beta = $ plasma energy/magnetic energy. We find

$$\beta\tau_E > \frac{48 \mu_0}{Q_\alpha} \frac{T_i^2}{\overline{\sigma v}\left(T_i\right)} \frac{1}{B^2} \quad \text{sec}$$

4. $\left(T_i^2/\overline{\sigma v}\right)$ has a minimum at about 15 keV. Thus,

$$\beta\tau_E > \frac{2.3}{B^2} \quad \text{sec}$$

5. *B* has a maximum from technological considerations:

$$B_{max} \sim 10 - 12\,\text{T}, \, B_{plas} \sim 5:$$

$$\beta\tau_E \underset{\sim}{>} \cdot 08$$

6. For fusion we need sufficiently high

a. τ_E : limited by transport, classical, neoclassical, anomalous. This is the regime of kinetic theory

b. β: limited by MHD equilibrium and stability considerations. This is the regime of MHD theory, ideal and resistive.

MHD Equilibrium Equations

1. Momentum:

$$\mathbf{J}\times\mathbf{B}=\nabla p$$

2. Maxwell:

$$\nabla\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{B} = \mu_0 \mathbf{J}$$

3. Note, consistent with the low frequency, long wavelength assumption, the displacement current is neglected in Ampere's law.

4. Note also that there is more information contained in the remaining moment equations. However, these equations are either trivially satisfied, or else give

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information about other quantities which can be calculated once the above set of equations is solved.

5. We will show that there is a wide degree of flexibility in finding solutions to the MHD equations; that is, the solutions are characterized by several free functions and boundary conditions which require transport theory, or some other physics to close the system.

Basic Properties of Ideal MHD Equilibrium

1. Virial theorem

2. Toroidal geometry

3. Flux surfaces and rotational transform

4. Basic problem of toroidal equilibrium

Virial Theorem

1. Question: Can the following configuration be held in equilibrium only by its own currents? If so, we could build a fusion reactor with modest, or no coils.
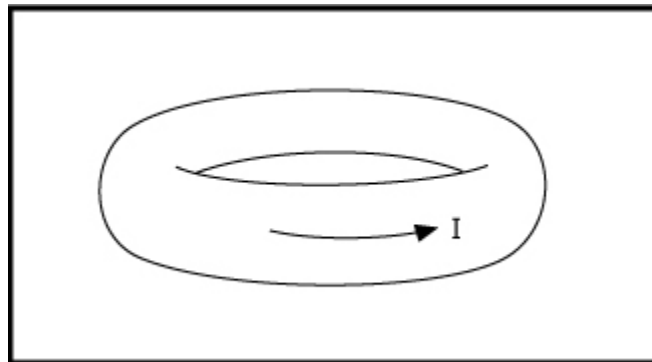
2. Answer: No!

3. Proof: Virial theorem

   a. $$\nabla p + \frac{1}{\mu_0} \nabla \times B \times B = 0$$

   $$\nabla \left(p + \frac{B^2}{2\mu_0} \right) - \frac{B \cdot \nabla B}{\mu_0} = 0$$

   $$\nabla \cdot (BB) = B \cdot \nabla B + B \nabla \cdot B$$

   $$\nabla \cdot \left[\left(p + \frac{B^2}{2\mu_0} \right) \ddot{I} - \frac{BB}{\mu_0} \right] = 0$$

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$$\nabla \cdot \overleftrightarrow{T} = 0 \quad \overleftrightarrow{T} = \begin{vmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{vmatrix}$$

$$p_\perp = p + B^2/2\mu_0$$
$$p_\parallel = p - B^2/2\mu_0$$

b. Integrate the following identity over the plasma volume

$$\nabla \cdot \left(r \cdot \overleftrightarrow{T} \right) = r \cdot \nabla \cdot \overleftrightarrow{T} + Trace \left(\overleftrightarrow{T} \right)$$

$$Trace \left(\overleftrightarrow{T} \right) = 2p_\perp + p_\parallel = 3p + \frac{B^2}{2\mu_0}$$

c. $\int_V \left(3p + \dfrac{B^2}{2\mu_0} \right) dr = \int_V \nabla \cdot \left(r \cdot \overleftrightarrow{T} \right) dr = \int_S dSn \cdot \left(r \cdot \overleftrightarrow{T} \right)$

$$= \int_S dS \left[(n \cdot r) \left(p + \frac{B^2}{2\mu_0} - (n \cdot b)(r \cdot b) \frac{B^2}{\mu_0} \right) \right]$$

d. Now assume the plasma can be confined by its own currents. Show that this leads to a contradiction.

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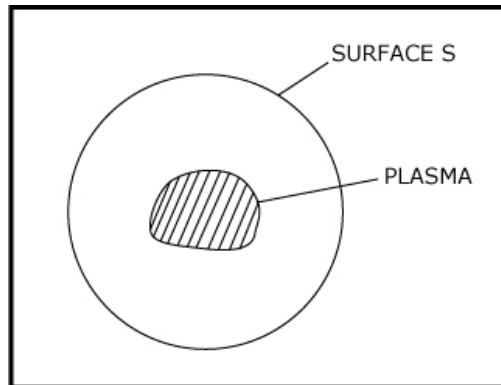
PLASMA

Choose *S* outside the plasma so that p(*S*)=0. Let $S \to \infty$: This is OK since there are no currents outside by assumption. For localized currents, the slowest decaying field scales as

$$\left| B(S) \right| \le \frac{K}{r^3} \text{ as } r \to \infty$$

e. $RHS = \int \underbrace{r^2 \sin\theta d\theta d\phi}_{dS} \left[r \dfrac{K^2}{r^6} - r \dfrac{K^2}{r^6} \right]$

$$= \int d\theta d\phi \frac{g(\theta, \phi)}{r^3} \to 0 \quad \text{as } r \to \infty$$

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f.　　$LHS = \int \left(3p + \dfrac{B^2}{2\mu_0} \right) dr \rightarrow$ finite, positive quantity

g. This is a contradiction. There must be current carrying coils outside the plasma. The surface integral must then include contributions from these surfaces.

Toroidal Geometry

1. Question: Why are most fusion configurations toroidal?

2. Answer: Avoid parallel end losses

3. Dominant loss mechanism is heat loss via thermal conduction

4. Heat loss is more severe along B than $\perp$ to B. The magnetic field confines particles in the $\perp$ direction.

5. Compare open ended and toroidal system
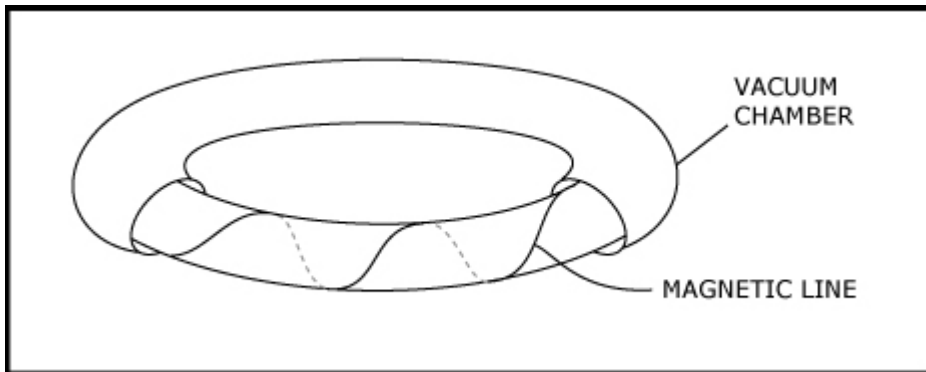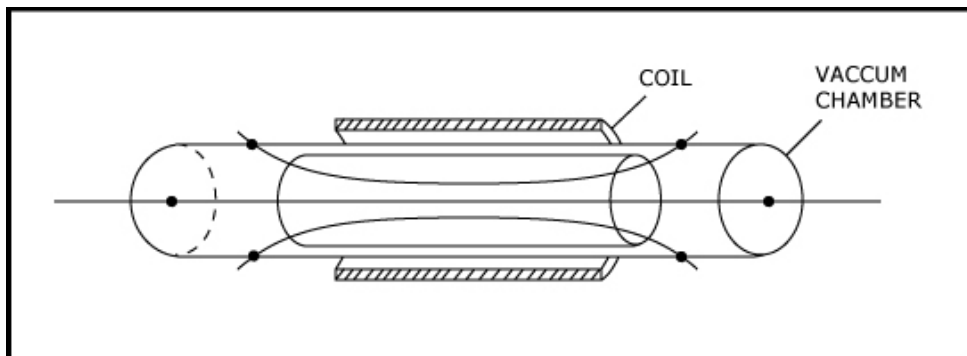
6. Open ended system: $\kappa_{\|e}$ dominant, electron end loss

7. Toroidal system: $\kappa_{\perp i}$ dominant, ion cross field transport

8. $\dfrac{\kappa_{\|e}}{\kappa_{\perp i}} \approx 1.12 \left(\dfrac{m_i}{m_e} \right)^{1/2} \left(\dfrac{T_e}{T_i} \right)^{5/2} (\Omega_i \tau_{ii})^2 \propto \dfrac{T^3 B^2}{n^2}$
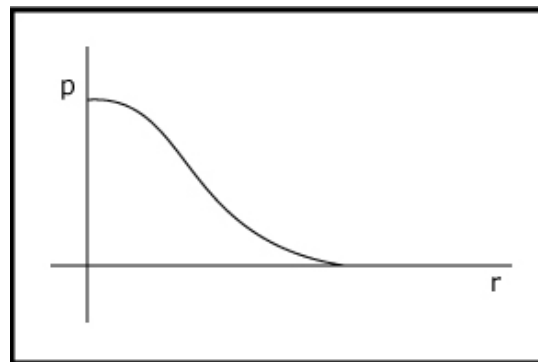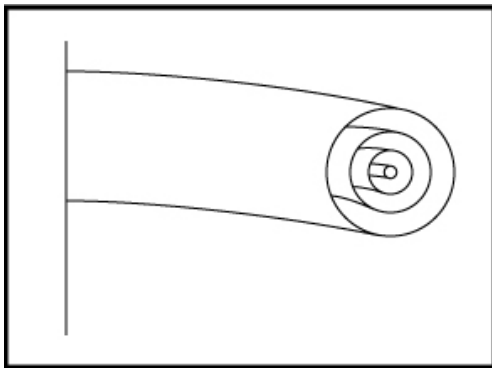
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9. For $T_i = 2\,\text{keV}$, $B = 5$ T, $n = 2 \times 10^{20} \text{m}^{-3}$

$$\frac{\kappa_{\|e}}{\kappa_{\perp i}} = 3 \times 10^{12}$$

10. Large difference in $\|$ and $\perp$ losses is the motivation for toroidicity

Flux Surfaces

1. In fusion configurations with confined plasmas the magnetic lines lie on a set of nested toroidal surfaces called flux surfaces.

2. This follows from equilibrium relation

$$\mathbf{B} \cdot \nabla p = 0$$

a. pressure is constant along a magnetic field line

b. magnetic lines lie in surfaces of constant pressure
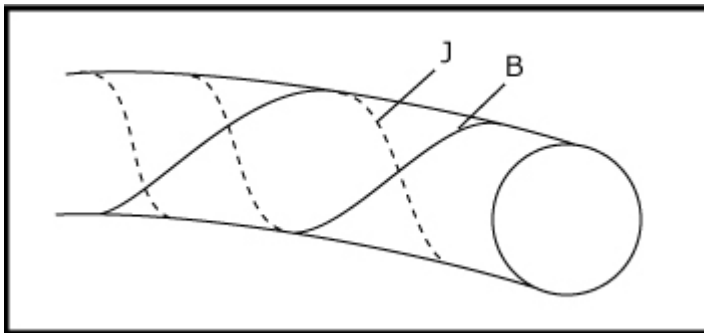
c. flux surfaces are surfaces of constant pressure

3. Similarly, the equilibrium relation

$$\mathbf{J} \cdot \nabla p = 0$$

implies that the current lines lie on surfaces of constant pressure. The current flows between flux surfaces and not across them
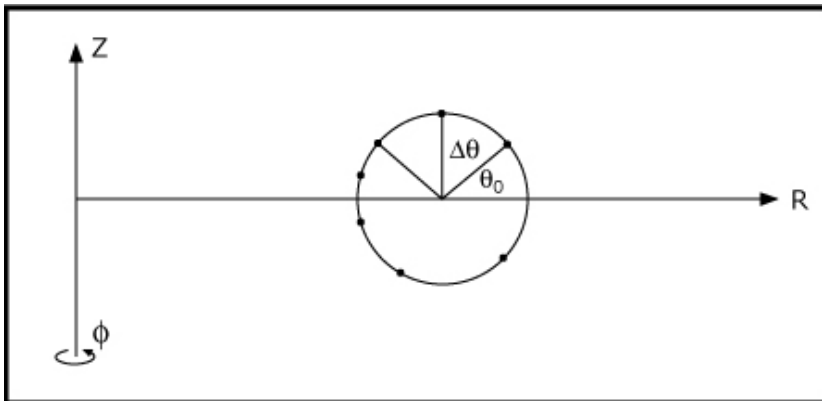
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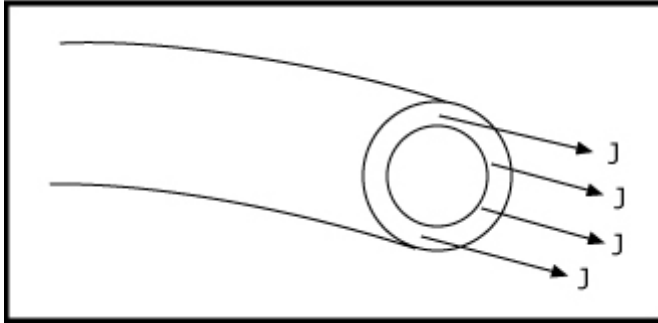
4. Does this imply that J and B are parallel?

Rotational Transform, Rational and Ergodic Surfaces

1. There are two classes of flux surfaces that must be distinguished: rational and ergodic.

2. The distinction is based upon rotational transform

3.

Rotational transform ι

$$\iota \equiv \underset{N \to \infty}{\text{Limit}} \quad \frac{1}{N} \sum_1^N \Delta\theta_n$$

4. Rotational transform is the *average* change in poloidal angle per single transit in the toroidal direction

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5. There are simpler ways to calculate ι that will be discussed in the future.

6. Rational surface

   $\iota/2\pi$ is rational fraction

   Magnetic lines close on themselves after a finite number of transits

   $$\frac{\iota}{2\pi} = \frac{n}{m}$$

7. Ergodic surface

   $\iota/2\pi$ is not rational

   magnetic lines eventually cover an entire surface

8. Stochastic volume: lines fill up a volume $\rightarrow$ ultra-poor confinement

9. Transform plays an important role in MHD equilibrium and stability

10. Often times the MHD safety is discussed in the literature

    $$q = \frac{2\pi}{\iota}$$

11. Note: for ergodic surfaces, the flux surfaces can be traced out by plotting contours of constant p or by following magnetic field line trajectories
    For rational surfaces, we must plot contours of constant p

12. Are there more rational or ergodic surfaces in general?

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