22.615 Final

Spring 2007

Part 2 – The Stability Problem

Six steps are required to carry out the stability analysis. (1) δW must be simplified using the high beta tokamak ordering. (2) A trial function will be specified for $\xi_r(r,\theta)$ in the plasma. It will contain three unknown amplitudes corresponding to a main harmonic plus one upper and one lower sideband. You will then need to evaluate ξ_θ, Q_{1r}, and $Q_{1\theta}$ in terms of ξ_r. (3) With this information the next step is to evaluate δW_F, which can be done numerically or analytically (if you have the stamina). δW_F should consist of a 3x3 quadratic form. (4) Next, in preparation for evaluating the vacuum energy, you need to evaluate the boundary conditions. (5) Knowing the boundary conditions it should then be straightforward to evaluate δW_V which again should consist of a 3x3 quadratic form. (6) Lastly, you can find the stability boundary by varying the equilibrium parameters, looking for those parameters where the lowest eigenvalue of the 3x3 matrix $\delta W = \delta W_F + \delta W_V$ is zero. This is the marginal stability transition.

To get things rolling I will do step (1) for you. The starting point for the analysis is the intuitive form of δW_F given by

$$\delta W_F = \int_P W_F d\mathbf{r}$$

where

$$W_F = \left|\mathbf{Q}_\perp\right|^2 + B^2\left|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\boldsymbol{\xi}\right|^2 - \frac{J_\parallel}{B}(\boldsymbol{\xi}_\perp^* \times \mathbf{B})\cdot\mathbf{Q}_\perp - 2(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*)$$

The stability analysis will focus on the most dangerous external ballooning-kink mode corresponding to a toroidal mode number $n=1$. Following the discussion

in class it then follows that for this mode the most unstable perturbations are incompressible.

Next is a slightly subtle point. Within the high beta tokamak ordering the largest term in the context of the ε expansion is due to magnetic compressibility. All other terms are smaller by ε^2. Please verify this yourself. Since magnetic compressibility is stabilizing we must choose $\nabla \cdot \xi_{\perp 0} = 0$ if instability is to occur. Furthermore, if we expand $\xi_\perp = \xi_{\perp 0} + \varepsilon \xi_{\perp 1} + \dots$ it then follows that the only appearance of $\xi_{\perp 1}$ is in the magnetic compressibility term. Since this is a positive definite term it is minimized by choosing $\nabla \cdot \xi_{\perp 1} = -2\xi_{\perp 0} \cdot \kappa$. The net result is that the magnetic compressibility term makes no contribution to δW_F and the remaining terms should be evaluated using $\xi_{\perp 0}$ as the trial function subject to the constraint $\nabla \cdot \xi_{\perp 0} = 0$.

With this insight we can simplify δW_F by noting that within the high beta tokamak ordering

$$\xi_{\perp 0} \approx \xi_r \mathbf{e}_r + \xi_\theta \mathbf{e}_\theta$$
$$\mathbf{Q}_{\perp 0} \approx Q_r \mathbf{e}_r + Q_\theta \mathbf{e}_\theta$$

and

$$W_F \approx |Q_r|^2 + |Q_\theta|^2 - \frac{J_\phi}{B}\left(\xi_\theta^* Q_r - \xi_r^* Q_\theta\right) - 2\left(\xi_r \frac{\partial p}{\partial r} + \frac{\xi_\theta}{r} \frac{\partial p}{\partial \theta}\right)\left(\xi_r^* \kappa_r + \xi_\theta^* \kappa_\theta\right)$$

with $\nabla \cdot \xi_\perp = 0$. For convenience all the zero subscripts have been suppressed.

We are now ready to proceed with part (2). Use the equilibrium flux function in the plasma ψ_{core} to evaluate the equilibrium magnetic field $B_\theta(r, \theta)$ and $B_r(r, \theta)$. *Note that there is a typo in the notes and in the textbook in the evaluation of* $B_r(r, \theta)$. *It should have a + sign in front and not a – sign. This follows from* $\nabla \cdot \mathbf{B} = 0$.

To evaluate δW_F assume that the trial function for the radial component of the displacement vector is given by

$$\xi_r(r,\theta) = \xi_m \rho^{m-1} e^{im\theta} + \xi_{m-1} \rho^{m-2} e^{i(m-1)\theta} + \xi_{m+1} \rho^m e^{i(m+1)\theta}$$

Here, the ξ_j are unknown variational coefficients and $\rho = r/a'$. In the analysis you will have to examine stability separately for various m values. Consider $m \geq 2$ as this is where most of the action takes place.

2. **Using this trial function derive analytic expressions for ξ_θ, Q_r, and Q_θ. This is slightly tedious.**

3. **Substitute these functions and evaluate δW_F. Your final expression should be of the form**

$$\delta W_F = \xi \cdot \mathbf{W}^{(F)} \cdot \xi$$

    where

$$\xi = \left[\xi_{m-1}, \xi_m, \xi_{m+1} \right]$$

$$\mathbf{W}^{(F)} = \begin{vmatrix} W_{11}^{(F)} & W_{12}^{(F)} & W_{13}^{(F)} \\ W_{21}^{(F)} & W_{22}^{(F)} & W_{23}^{(F)} \\ W_{31}^{(F)} & W_{32}^{(F)} & W_{33}^{(F)} \end{vmatrix}$$

    The matrix $\mathbf{W}^{(F)}$ should be symmetric and all its elements should be real. This step involves a substantial part of the work. You may want to evaluate the elements numerically. Make sure you only keep the leading order, non-vanishing contributions in the ε expansion.

Turn now to the contribution from the force free plasma and vacuum region. Following the procedure described in class assume initially that the entire region

beyond the core is a vacuum region; that is, ignore the force free plasma. Thus write $\hat{B}_1 = \nabla \Phi$ with Φ satisfying $\nabla^2 \Phi = 0$.

4. **(a) Evaluate the boundary condition at the plasma core-vacuum interface. If the trial function for the displacement has three harmonics, how many harmonics does Φ have? Hint: It is not three.**
   (b) Looking ahead to the inclusion of the force free plasma region, assume that for each harmonic m', the plasma core, for simplicity, is surrounded by a circular, concentric, perfectly conducting wall located at $r = b_{m'}$ where $b_{m'}$ will be specified shortly. What is the boundary condition for each harmonic at $r = b_{m'}$?

5. **Solve for the potential Φ using the high beta tokamak ordering. Your final result should be of the form**

$$\delta W_V = \xi \cdot \mathbf{W}^{(V)} \cdot \xi$$

   where

$$\mathbf{W}^{(V)} = \begin{vmatrix} W_{11}^{(V)} & W_{12}^{(V)} & W_{13}^{(V)} \\ W_{21}^{(V)} & W_{22}^{(V)} & W_{23}^{(V)} \\ W_{31}^{(V)} & W_{32}^{(V)} & W_{33}^{(V)} \end{vmatrix}$$

   and $\mathbf{W}^{(V)}$ is real and symmetric. You should be able to evaluate the matrix elements analytically. Hint: I do not expect to see a single Bessel function at any point in the calculation.

6. **Now for the interesting part. Form the complete potential energy matrix $\mathbf{W} = \mathbf{W}^{(F)} + \mathbf{W}^{(V)}$. A straight forward minimization shows that the minimizing eigenfunction satisfies $\mathbf{W} \cdot \xi = 0$. Marginal stability is**

determined by examining the eigenvalues of this 3x3 matrix. When the plasma parameters are chosen so that the lowest eigenvalue is zero, this corresponds to the marginal stability transition. The eigenvalues can easily be found numerically using standard a linear algebra package.

To carry out the analysis you will need to evaluate $b_{m'}$. Do this as follows. For any set of plasma parameters evaluate $q_{a'}$ and q_a. Consider next the value of m' for the harmonic under consideration. If m' is such that $q_{a'} < q_a < m'$ then the corresponding resonant surface lies in the vacuum region. For this case choose $b_{m'} = \infty$ - the wall is at infinity. For $q_{a'} < m' < q_a$ the resonant surface lies in the force free region. Assuming for simplicity a quadratic dependence on the safety factor

$$q = q_{a'} + (q_a - q_{a'})\left(\frac{r^2 - a'^2}{a^2 - a'^2}\right)$$

choose $r = b_{m'}$ so that $q = m'$. In other words

$$b_{m'}^2 = a'^2 + \left(a^2 - a'^2\right)\left(\frac{m' - q_{a'}}{q_a - q_{a'}}\right)$$

For the last case, $m' < q_{a'} < q_a$, the resonant surface appears in the core. For this situation choose $b_{m'} = \infty$ and for simplicity do not make any alterations to the plasma core trial function.

Now vary the plasma parameters ζ_1 and α to determine the marginal stability boundary in the form of a curve of β_t / ε vs $1/q_*$. You will need to do this separately for a few values of m to find the worst case. Superimpose your results on the equilibrium curve of β_t / ε vs $1/q_*$.

What is the highest stable value of β_t / ε? What is the corresponding value of q_*? How does this compare with the Troyon limit?

YOU ARE NOW AN MHD EXPERT!!