INTRO. PLASMAS I - FALL 2002
FINAL EXAM SOLUTIONS

1) SHORT ANSWER

a) $\omega_{pe}^2 = 4\pi n_e e^2/m_e$; $\omega_{pi}^2 = 4\pi n_i e^2 Z_i^2/m_i$

b) $\omega_{ci} = eZ_i B/m_i c$ $\qquad = 4\pi n_e Z_i e^2/m_i$

c) $\lambda_e^2 = T_e/4\pi n_e e^2$ $\qquad$ (FOR $n_i Z_i = n_e$ PLASMA)

$\lambda_i^2 = T_i/4\pi n_i Z_i^2 e^2$

d) TRUE ; e) TRUE ; f) FALSE ; g) TRUE ; h) TRUE

i) FALSE

2) WAVE ENERGY & DISSIPATION

Since ϵ is analytic it can be Taylor series expanded about $\omega = \omega_r$ for small γ/ω_r. (The derivative does not depend on direction in complex plane for an analytic function).

$$\epsilon(k, \omega_r + i\gamma) \doteq \epsilon_R(k, \omega_r) + i\epsilon_I(k, \omega_r) + i\gamma \frac{\partial \epsilon_R}{\partial \omega} - \underbrace{\gamma \frac{\partial}{\partial \omega} \epsilon_I}_{\text{2ND. ORDER (IGNORE)}}$$

Equating REAL and IMAGINARY parts separately:

$\epsilon_R(k, \omega_r) = 0$ GIVES $\omega_r = \omega_r(k)$ DISPERSION RELATION

(Real frequency comes from REACTIVE RESPONSE)

$$\gamma = -\frac{\epsilon_I(k,\omega_r)}{\partial \epsilon_R / \partial \omega_r} \Rightarrow$$

$$\frac{\gamma}{\omega} = -\frac{\epsilon_I(k,\omega_r)}{\omega \partial \epsilon_R / \partial \omega |_{\omega=\omega_r}} = -\frac{\text{DISSIPATION}}{\text{WAVE ENERGY}}$$

The growth (or damping) rate in dimensionless terms is negative the ratio of DISSIPATION (ϵ_I) to WAVE ENERGY ($\omega\, \partial\epsilon_R/\partial\omega$). Positive dissipation implies damping (for positive energy modes) & negative dissipation leads to growth.

FROM Eq. (1)

$$\epsilon(k,\omega_r) \to 1 - \frac{\omega_{pe}^2}{k^2} P \int du \frac{1}{u - \omega_r/k} \frac{\partial F}{\partial u} - i\pi \frac{\omega_{pe}^2}{k^2} \frac{\partial F}{\partial u}\Big|_{u=\omega_r/k}$$

$$\epsilon_I = -\pi \frac{\omega_{pe}^2}{k^2} \frac{\partial F}{\partial u}\Big|_{u=\omega_r/k}$$

3) ELECTRON BEAM INSTABILITIES

Evaluate labels:

$V_e = V_b = \sqrt{2T_e/m_e} = \sqrt{\frac{2\,\text{keV}}{m_e c^2}}\; c$

$v_{b1} = \sqrt{20\,\text{keV}/m_e}$; $v_{b2} = \sqrt{200\,\text{keV}/m_e}$

Waves are high phase velocity e-PLASMA OSCILLATIONS with $\omega_r \doteq \omega_{pe}$ (since $\omega/k \gg v_e$ & $n_b/n_p \ll 1$ for beams)

There are 2 UNSTABLE BANDS where distribution has positive slope:

BAND 1: $V_{m1} < \frac{\omega}{k} < v_{b1}$; $\frac{\omega}{k} \gg V_e$

BAND 2: $V_{m2} < \frac{\omega}{k} < v_{b2}$; $\frac{\omega}{k} \gg v_{b1}, V_e$

WAVE ENERGY

$\epsilon_R \doteq 1 - \frac{\omega_{pe}^2}{\omega^2} \Rightarrow \omega\, d\epsilon_R / d\omega \doteq 2$ DIMLESS WAVE ENERGY

This can be argued on physical grounds or shown via expansion:

$$\epsilon_R = 1 - \frac{\omega_{pe}^2}{k^2} P\int du\, \frac{k}{\omega} \frac{-1}{1-\frac{uk}{\omega}} \frac{\partial F}{\partial u} = 1 + \frac{\omega_{pe}^2}{k^2}\frac{k}{\omega}\int du \left(1 + \frac{uk}{\omega} + \frac{u^2k^2}{\omega^2}\right)\frac{\partial F}{\partial u}$$

& integrate by parts $\Rightarrow$ this survives

$$\cong 1 - \frac{\omega_{pe}^2}{\omega^2} + \text{BOHM-GROSS CORRECTIONS}$$

THE (WEAK) DISSIPATION is simply

$$\epsilon_I = -\pi \frac{\omega_{pe}^2}{k^2} \frac{\partial F}{\partial u}\Big|_{u=\omega_r/k}$$

F is a sum of Maxwellian's:

$$F = \frac{1}{\sqrt{\pi}\, v_e}\Bigg[\Big(1 - \underbrace{\frac{n_{b1}+n_{b2}}{n}}_{\text{SMALL}}\Big) e^{-v^2/v_e^2} + \frac{n_{b1}}{n} e^{-(v-U_{b1})^2/v_e^2} + \frac{n_{b2}}{n} e^{-(v-U_{b2})^2/v_e^2}\Bigg]$$

Growth rates will be determined from $\partial F/\partial u$ & wavenumber, k, giving resonant phase velocity: $v_\phi = \omega_{pe}/k$.

MOST UNSTABLE MODES:

Here we need only find maximum in $\frac{\partial F}{\partial u}$ for each beam:

$$\frac{\partial^2 F}{\partial u^2} = 0 = -\frac{2}{v_e^2}\Big(1 - \frac{2(v-U_{b1})^2}{v_e^2}\Big)\frac{n_{b1}}{n} e^{-(v-U_{b1})^2/v_e^2}$$

(FOR BEAM 1)

$$\Rightarrow v_\phi(\text{MAX } \gamma) = U_{b1} - \frac{v_e}{\sqrt{2}}$$

$$\text{OR } \boxed{k_{MAX} = \frac{\omega_{pe}}{U_{b1} - v_e/\sqrt{2}}} \cong \frac{\omega_{pe}}{U_{b1}}$$

$$\left.\frac{\partial F}{\partial u}\right|_{MAX} = \sqrt{\frac{2}{\pi}}\,\frac{n_{b1}}{n v_e^2}\,e^{-1/2} = \sqrt{\frac{2}{\pi e}}\,\frac{n_{b1}}{n v_e^2}$$

MAXIMUM GROWTH RATE IN BAND 1 IS THEREFORE:

$$\frac{\gamma_{MAX}}{\omega_{pe}} \simeq -\frac{\epsilon_I}{2} = \sqrt{\frac{\pi}{2e}}\,\frac{n_{b1}}{n v_e^2}\,\frac{\omega_{pe}^2}{k_{MAX}^2} \cong \sqrt{\frac{\pi}{2e}}\,\frac{n_{b1}}{n}\,\frac{U_{b1}^2}{v_e^2}$$

SIMILARLY FOR BAND 2 ($1 \to 2$, same formula)

Since $\gamma \propto U_b^2$ the wave resonant at maximum gradient of highest energy beam will have highest growth rate. With $\hat{\phi} \propto e^{\gamma t}$ this mode will become dominant as t gets large (in the linear regime).

4) ION ACOUSTIC WAVES

LIMITING FORMS OF RESPONSE FOR: $v_i \ll \frac{\omega}{k} \ll v_e$

ELECTRONS

$$\frac{\omega}{k v_e} Z\left(\frac{\omega}{k v_e}\right) \to O\left(\left(\frac{\omega}{k v_e}\right)^2\right) \to 0$$

$$\chi_e \cong \frac{1}{k^2 \lambda_e^2} \qquad \text{ADIABATIC RESPONSE}$$

GEOMETRIC SERIES FOR $\frac{1}{1-t/\xi}$

<u>IONS</u> $\quad \xi \equiv \frac{\omega}{k v_i} \gg 1$

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int dt\, e^{-t^2} \left(\frac{-1}{\xi}\right)\left(1 + \frac{t}{\xi} + \frac{t^2}{\xi^2} + \cdots\right)$$

$$\cong -\frac{1}{\xi}\left(1 + \frac{1}{2\xi^2}\right)$$

$$\Rightarrow 1 + \xi Z(\xi) = -\frac{1}{2\xi^2} = -\frac{k^2 v_i^2}{2\omega^2} = -\frac{k^2 T_i}{\omega^2 m_i}$$

combining back to give ϵ:

$$\epsilon \cong 1 + \frac{1}{k^2\lambda_e^2} + \frac{1}{k^2\lambda_i^2}\left[-\frac{k^2 T_i}{\omega^2 m_i}\right]$$

$$= 1 + \frac{1}{k^2\lambda_e^2}\left(1 - \frac{k^2 c_s^2}{\omega^2}\right) \; ; \; c_s^2 = T_e/m_i$$

For $k^2\lambda_e^2 \ll 1$, we may neglect the "1" term in dielectric to give:

$$\omega^2 = k^2 c_s^2.$$

Note that "1" corresponds to LHS of Poisson's equation and ignoring it implies, $\hat{n}_e \cong \hat{n}_i$. This is QUASINEUTRALITY. It is a result of Debye shielding of ion fluctuations by electrons in the long wavelength, $k\lambda_e \ll 1$, limit.

5) ION-ELECTRON COLLISIONS

SINCE COLLISIONS CONSERVE ENERGY WE KNOW THAT:

$$\int d^3v\left(\frac{1}{2}m_e v^2 C_{ei}(f_e,f_i) + \frac{1}{2}m_i v^2 C_{ie}(f_i,f_e)\right)=0$$

The first term represents the rate of change of electron energy due to e- collisions with ions. The second term represents the rate of change of ion energy due to ion collisions with electrons. These two terms must be equal & opposite ⇒

$$\int d^3v\, \frac{1}{2}m_i v^2 C_{ie}(f_i,f_e) = \nu_{ei}\frac{m_e}{m_i}\sqrt{\frac{2}{\pi}}\,n(T_e-T_i)$$

MAXWELLIANS

If $T_e > T_i$ the ions are heated until, $T_i = T_e$ or equilibrium is achieved.

6) THERMAL EQUILIBRIUM

a) METHOD OF LAGRANGE MULTIPLIERS:

PERFORM FUNCTIONAL VARIATION ON

$\Lambda = S + \alpha n + \beta E$ with α, β MULTIPLIERS

$$\delta\Lambda = \int d^3v \left[\delta(f \ln f) + \alpha\,\delta f + \beta \tfrac{1}{2} m v^2 \delta f \right]$$

$$= \int d^3v\, \delta f \left[\ln f + 1 + \alpha + \beta \tfrac{1}{2} m v^2 \right]$$

$$\delta\Lambda = 0 \quad \forall\, \delta f \Rightarrow 0 = \ln f + 1 + \alpha + \beta \tfrac{1}{2} m v^2$$

$$\boxed{f = \exp\left(1 - \alpha + \beta \tfrac{1}{2} m v^2\right)}$$

α & β can be calc'd from n & E

DEFINE $E \equiv \frac{3}{2} n T \Rightarrow f = \dfrac{n}{(\pi v_{th}^2)^{3/2}} \exp\left(-\frac{1}{2}\frac{m v^2}{T}\right)$

b) ENTROPY INCREASES MONOTONICALLY IN TIME UNTIL, $f = f_{MAX}$, AT WHICH POINT ENTROPY IS MAXIMUM.

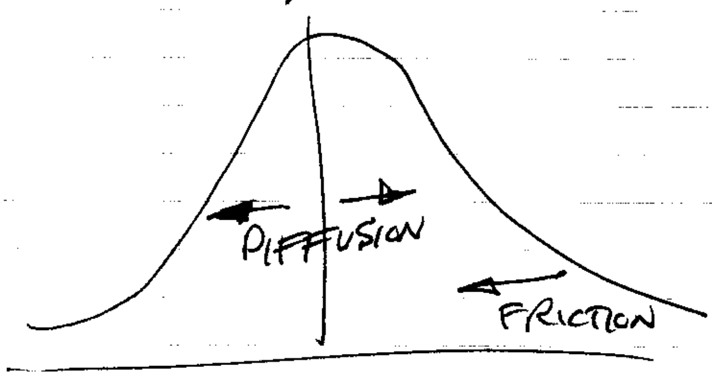
c) $C_{ei}^{E}(f_e) = 0 = (h) \dfrac{\partial}{\partial v} \underbrace{\left[f_e + \dfrac{T_i}{m_e} \dfrac{1}{v} \dfrac{\partial}{\partial v} f_e \right]}$

$= \text{CONST.} = 0$ since

$f_e, \frac{\partial f_e}{\partial v} \to 0$ as $v \to \infty$

$$0 = f_e + \frac{T_i}{m_e}\frac{1}{v}\frac{\partial}{\partial v} f_e \Rightarrow f_e = \text{CONST.} \times \exp\left(-\frac{m_e v^2}{2T_i}\right)$$

The term $\propto \frac{\partial f_e}{\partial v}$ represents a friction in velocity space, tending to drag particles to zero velocity. The term $\propto \frac{\partial}{\partial v}\frac{1}{v}\frac{\partial}{\partial v} f_e$ represents a diffusion process causing distribution to spread in velocity space:



The Maxwellian distribution is the shape that balances these processes.