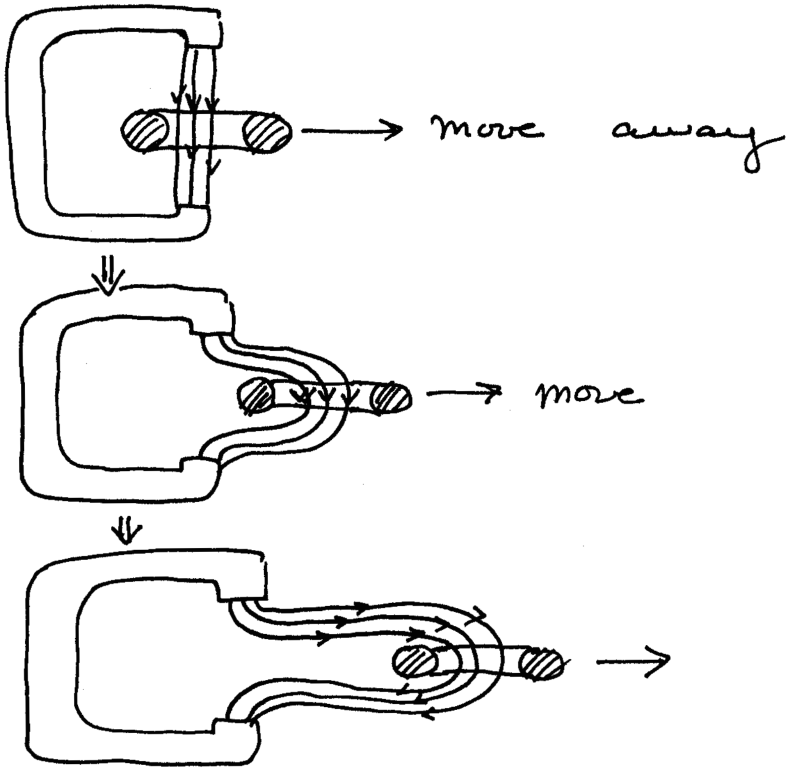
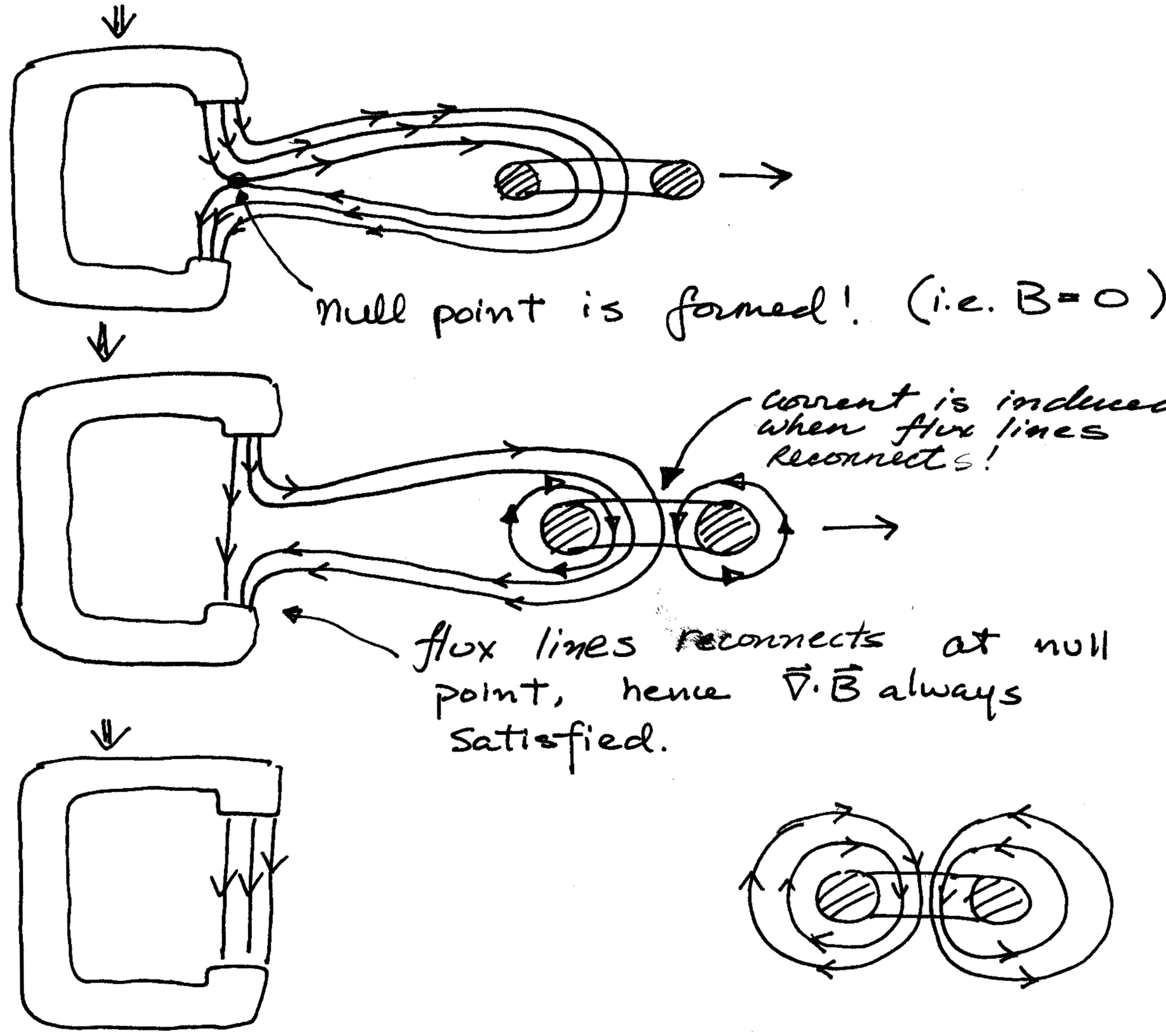
PS #2 Answer Key 22.611J U. Tang

1)



· Notes:
→ Reconnection occurs only w/ resistivity (you'll learn about that later!)
→ the flux through the torus is constant thru the whole process! (Current is induced as required.)



2. Polarization Drift

We now've $v_{E\times B} = v_{E\times B}(t)$

Using conservation of energy,

$$\frac{d}{dt}\left(\frac{1}{2} m \vec{v}_{E\times B}^{\,2}\right) = \vec{v}_{pol} \cdot q\, \vec{E}_\perp(t)$$

$$\vec{v}_{E\times B} = \frac{\vec{E}_\perp \times \vec{B}}{B^2}$$

$$v_{E\times B}^2 = \frac{E_\perp^2}{B^2}$$

then.

$$\frac{d}{dt}\left(\frac{1}{2} m \frac{E_\perp^2}{B^2}\right) = \frac{1}{\cancel{2}}\frac{m}{B^2}\,\cancel{2}\cancel{\vec{E}_\perp}\cdot\frac{d\vec{E}_\perp}{dt} = \vec{v}_{pol}\cdot q\,\cancel{\vec{E}_\perp}(t)$$

then $$\boxed{\vec{v}_{pol} = \frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt} = \frac{1}{\Omega B}\frac{d\vec{E}_\perp}{dt}}$$

→ that's the quick way ....

→ the rigorous way uses gyro-average theory ⟶

2) Polarization Drift (Con't)

- Let's start w/ basic EOM

$$m\frac{d\vec{v}}{dt} = q(\vec{E}(t) + \vec{v}\times\vec{B})$$

- Choose $\vec{B} = B_0\hat{z}$ & $\vec{E} = E(t)\hat{x}$

then we've

$$(1) \quad m\frac{dv_x}{dt} = qE_x(t) + qv_y B_0$$

$$(2) \quad m\frac{dv_y}{dt} = -qv_x B_0$$

taking a derivative of (1) & (2) & substituting, we've

$$m\frac{d^2v_x}{dt^2} = q\frac{\partial E_x}{\partial t} + q\left(-\frac{q}{m}v_x B_0\right)B_0$$

&

$$m\frac{d^2v_y}{dt^2} = -q\left(\left(\frac{q}{m}\right)(E_x(t) + v_y B_0)\right)B_0$$

Rewriting,

$$(3) \quad \frac{d^2v_x}{dt^2} = \frac{\Omega}{B_0}\frac{\partial E_x}{\partial t} - \Omega^2 v_x$$

$$\frac{d^2v_y}{dt^2} = -\Omega\left(\frac{\Omega}{B_0}E_x + \Omega v_y\right)$$

$$(4) \quad \frac{d^2v_y}{dt^2} = -\frac{\Omega^2}{B_0}E_x - \Omega^2 v_y$$

2) Con't

Consider $\vec{v} = \vec{v}_0 + \vec{v}_1 + \vec{v}_2$,

where v_0 is the gyro-motion
v_1 the standard $E \times B$ drift
v_2 the polarization drift

then,

$$\frac{d^2(\vec{v}_0 + \vec{v}_1 + \vec{v}_2)}{dt^2} = \underbrace{\frac{\Omega}{B_0}\frac{\partial \vec{E}_x}{\partial t}}_{2^{nd}\text{order}} - \underbrace{\frac{\Omega^2}{B_0}}_{1^{st}\text{ order}} E_x \hat{y} - \Omega^2(\vec{v}_0 + \vec{v}_1 + \vec{v}_2)$$

so, we've got

- 0th order $\frac{d^2\vec{v}_0}{dt^2} = -\Omega^2 \vec{v}_0$, which is zeroth order gyro-motion

  (if u take the gyro-averaged $\frac{d^2\langle\vec{v}_0\rangle}{dt^2} = 0$ so $\langle\vec{v}_0\rangle = 0$!) ↑ Guilding center

- 1st order $\frac{d^2 v_{1y}}{dt^2} = -\frac{\Omega^2}{B_0} E_x - \Omega^2 v_{1y}$

  taking the gyro-average, we've

  $$\frac{d^2\langle v_{1y}\rangle}{dt^2} = -\frac{\Omega^2}{B_0}\langle E_x\rangle - \Omega^2\langle v_{1y}\rangle$$

  then, $\frac{d^2\langle v_{1y}\rangle}{dt^2} << -\Omega^2\langle v_{1y}\rangle$

  so

  $$v_{1y} = -\frac{\langle E_x\rangle}{B_0} \Rightarrow \text{which is just the } \frac{\vec{E}\times\vec{B}}{B^2} \text{ drift}$$

- 2nd order $\frac{d^2\langle v_{2x}\rangle}{dt^2} = \frac{\Omega}{B_0}\frac{\partial\langle E_x\rangle}{\partial t} - \Omega^2\langle v_{2x}\rangle$

  again $\frac{d^2\langle v_{2x}\rangle}{dt^2} \ll -\Omega^2\langle v_{2x}\rangle$,

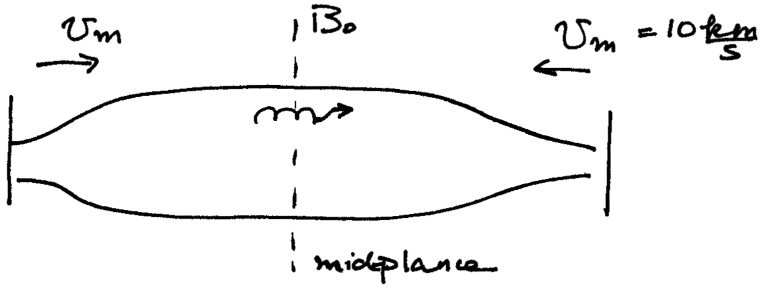
  so

  $$+\Omega^2\langle v_{2x}\rangle = \frac{\Omega}{B_0}\frac{\partial\langle E_x\rangle}{\partial t}$$

  $$\boxed{\langle v_{2x}\rangle = \frac{1}{\Omega B_0}\frac{\partial\langle E_x\rangle}{\partial t}}$$

  the polarization drift!
  In general
  $\vec{v}_2 = \frac{1}{\Omega B_0}\frac{\partial\vec{E}}{\partial t}$

3) Fermi Acceleration of Cosmic Rays



- Let i & f represent the initial and final states (before & after acceleration)

a) • Since μ is constant both in time and space, we first see for velocities at the midplane:

$v_{\perp i} = v_{\perp f}$ at midplane, since $B_{0i} = B_{0f}$
$(\mu_i = \mu_f)$

- Now, let's examine the situation of the μ constant in space at t_{final}.

So we've for the loss cone condition:

$$\frac{B_{0f}}{B_{mf}} = \sin^2\theta_m = \frac{1}{R_m} = \frac{1}{5} = \frac{v_{\perp 0f}^2}{v_{\perp mf}^2}$$

- Using conservation of energy at t_{final}:

$$v_{\perp 0f}^2 + v_{\parallel 0f}^2 = v_{\perp mf}^2$$

then,

$$\frac{1}{5} = \frac{v_{\perp 0f}^2}{v_{\perp 0f}^2 + v_{\parallel 0f}^2}$$

- Using $v_{\perp 0f}^2 = v_{\perp 0i}^2$

$$\frac{1}{5} = \frac{v_{\perp 0i}^2}{v_{\perp 0i}^2 + v_{\parallel 0f}^2} = \frac{1}{1 + \frac{v_{\parallel 0f}^2}{v_{\perp 0i}^2}}$$

hence, $\frac{v_{\parallel 0f}^2}{v_{\perp 0i}^2} = 4 \Rightarrow v_{\parallel 0f} = 2 v_{\perp 0i}$

So Now let's look at the Energies at the mid-plane

$$E_i = \frac{1}{2}m\left(v_{\perp 0i}^2 + v_{\parallel 0i}^2\right) = \frac{2}{2}m\left(v_{\perp 0i}^2\right) = m v_{\perp 0i}^2$$

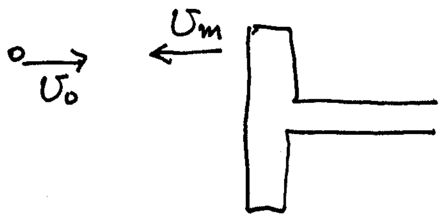
$$E_f = \frac{1}{2}m\left(v_{\perp 0f}^2 + v_{\parallel 0f}^2\right) = \frac{1}{2}m\left(v_{\perp 0i}^2 + 4 v_{\perp 0i}^2\right)$$

so we've

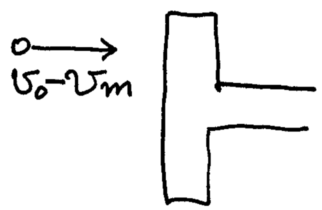
$$E_i = m v_{\perp 0i}^2 \quad , \quad E_f = \frac{5}{2} m v_{\perp 0i}^2$$

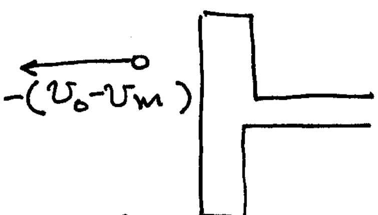
so, since $\boxed{E_i = 1\,keV, \quad E_f = 2.5\,keV}$

b) → We've to first determine the change in velocity per bounce:



In the frame of the piston, we've





• Before Collision • After collision

(The collision is elastic and the mass of the piston >> m_{proton})

so, in the lab frame,

$$v_f = v_f' + v_m = -(v_0 - v_m) + v_m = -v_0 + 2v_m$$

hence, the change in velocity is $2|v_m|$

→ Now let's figure out how many bounces we need for achieving $v_{\parallel 0f}$. since the change in momentum for each bounce is $\Delta \vec{p}_{\parallel} = 2m|v_m|$. So, for N bounces, we've

$$P_{\parallel f} = P_{\parallel 0i} + N\Delta p$$

b) Con't

then,

from pt. a

$$\frac{P_{\parallel of} - P_{\parallel oi}}{\Delta p} = N = \frac{v_{\parallel of} - v_{\parallel oi}}{2 v_m} = \frac{2 v_{\perp oi} - v_{\perp oi}}{2 v_m} = \frac{1}{2} \frac{v_{\perp oi}}{v_m}$$

$$N = \frac{1}{2} \frac{3.1\times10^5 \text{ m/s}}{(10 \text{ km/s})} \quad \left(1 \text{ keV} = \frac{1}{2} m_p (v_{\perp oi} + v_{\parallel oi}) \text{ w/ } v_{\parallel oi} = v_{\perp oi} \right)$$

$N \cong 15.5$ ~~times~~! bounces

- Now, assume that $L \gg v_m \tau$, where τ is the time it takes to reach $v_{\parallel of}$ & escape. (so $L \sim$ constant)

- Now, let's determine $\bar{v}_{\parallel o}$ (average)

$$\bar{v}_{\parallel o} = \frac{1}{2}(v_{\parallel oi} + v_{\parallel of}) = \frac{1}{2}(v_{\perp oi} + 2 v_{\perp oi})$$

$$= \frac{3}{2} v_{\perp oi} \cong 4.65\times10^5 \frac{m}{s}$$

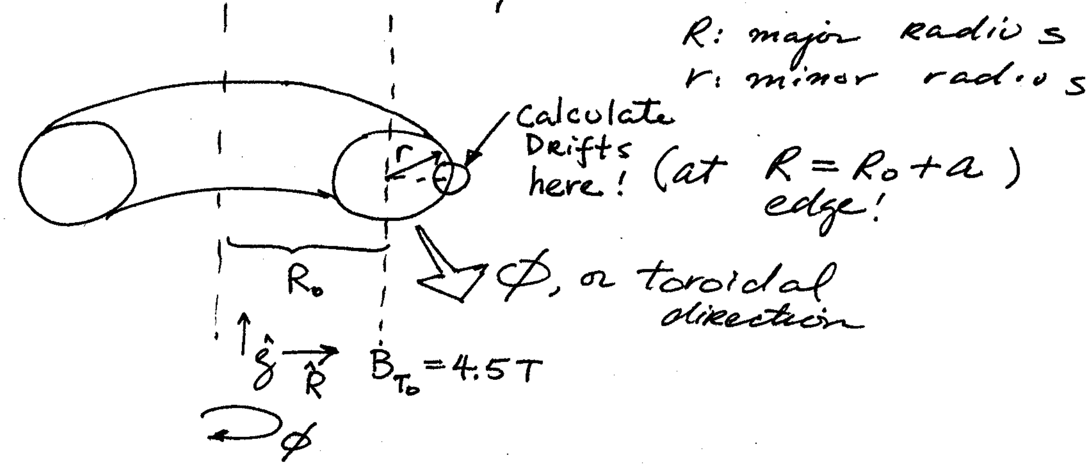
- So now determine the time it takes to do 15.5 bounces!

- So $\Delta x_{total} = 15.5 L$ &

$$\boxed{\tau = \frac{\Delta x_{total}}{\bar{v}_{\parallel o}} = \frac{15.5 L}{4.65\times10^5} = 3.33\times10^{-5} \left(\frac{s}{m}\right) L}$$

4) Numbers for Drifts

Tokamak Geometry



→ 1st thing we've to do is calculate B_T at the edge.

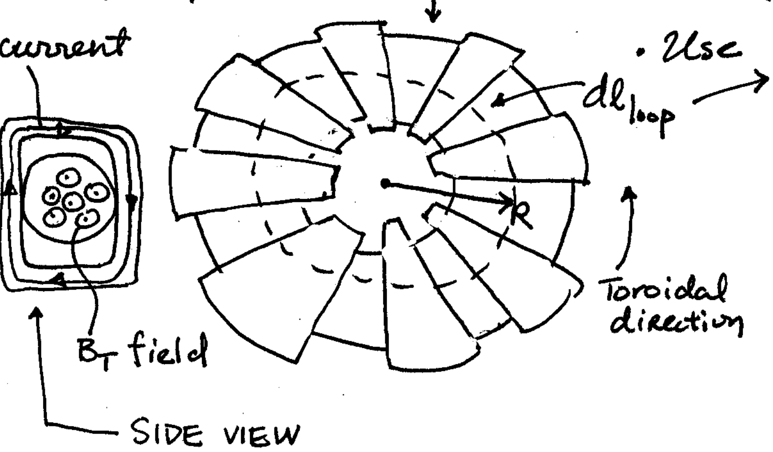
→ from class:

$$B(R) = \frac{B_0 R_0}{R} = \frac{4.5T(.66m)}{(.66m + .21m)} = 3.41\,T$$

→ Derivation from Maxwell equation:

- We know $B \sim B_T = 4.5T$ at $r = 0$
- Using the low-β assumption, we can use Maxwell's equation to solve for $B_T(\hat{R})$, since we know B_T is created through a set of uniformed coils:

Bottom VIEW OF TOKAMAK:



- Use $\mu_0 \int_S J\,dA = \int_C B \cdot d\bar{l}$ (Ampère's Law)

then,

$$\mu_0 I = B_T 2\pi R$$

(I is total current from coils)

$$\boxed{B_T = \frac{\mu_0 I}{2\pi R} \quad (\text{in SI}) \qquad = \frac{2I}{Rc} \quad (\text{in cgs})}$$

c = speed of light

4) Cont

$\vec{E}\times\vec{B}$ drifts:

In SI, $\vec{v}_{E\times B} = \dfrac{\vec{E}\times\vec{B}}{B^2} = \dfrac{E_r B_T(R+a)}{B_T(R+a)^2} = \dfrac{10^5 V/m}{3.41 T}$

$$\boxed{\vec{v}_{E\times B} = 2.9\times10^4 \frac{m}{s}}$$

Independent of q, m, or $v_\perp$!

$\vec{\nabla}B \perp \vec{B}$ drift

In SI,

$$v_{\nabla B} = \pm\frac{1}{2} v_\perp r_L \frac{\vec{B}\times\vec{\nabla}B}{B^2} \quad \text{where } r_L = \frac{m v_\perp}{qB}$$

[Note: $\vec{B}\times\vec{\nabla}B$
$\begin{vmatrix} \hat{R} & \hat{\phi} & \hat{z} \\ 0 & B_T & 0 \\ (\nabla B)_R & 0 & 0 \end{vmatrix}$
$= \hat{z}(-B_T(\vec{\nabla}B)_R)$
$= +\dfrac{B_T B_0 R_0}{R^2}$]

$$v_{\nabla B} = \pm\frac{1}{2}\frac{v_\perp^2 m}{qB^3}\vec{B}\times\vec{\nabla}B$$

$$v_{\nabla B} = \pm\frac{1}{2}\frac{v_\perp^2 m}{qB_T^3}\left(\hat{z}\,\frac{B_T B_{0T} R_0}{R^2}\right)$$

$$v_{\nabla B} = \pm\frac{1}{2}\frac{2T\,B_{0T}R_0}{qB_T^2R^2}\hat{z} \quad \left(\text{since } \tfrac{1}{2}m\bar{v}_\perp^2 \sim T \ \&\ \tfrac{1}{2}m\bar{v}_\parallel^2 \sim \tfrac{1}{2}T\right)$$

2 DOF

1 DOF

total: $\frac{3}{2}T = v_{th}$

then, in dimensionless form,

$$\boxed{\left|\frac{v_{\nabla B\perp B}}{v_{th_i}}\right| = \frac{T B_0 R_0}{qB_T^2 R^2 v_{th_i}}\hat{z} = \frac{T}{qB_T R v_{th_i}}\hat{z}}$$

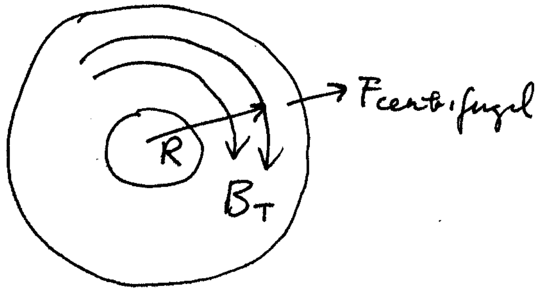
for our problem

$v_{th_i} = \left(\dfrac{3T}{m}\right)^{\frac{1}{2}} = 7.6\times10^5 \dfrac{m}{s}$ for hydrogen ions then

$$\boxed{\left|\frac{v_{\nabla B \perp B}}{v_{thi}}\right| = \frac{(2000\,eV)(\cancel{1.6\times10^{-19}}\,J/eV)(4.5\,T)(.66\,m)}{(\cancel{1.6\times10^{-19}}\,C)(3.41\,T)^2(.66\,m + .21\,m)^2\,v_{thi}} = 8.9\times10^{-4}}$$

Curvature Drift

TOP VIEW



In SI Units

- $\vec{v}_{cur} = \dfrac{m v_{\parallel}^2}{q B^2}\,\dfrac{\vec{R}\times\vec{B}}{R^2}$
- $\vec{R}\times\vec{B} = \hat{z}\,(R B_T)$
- $v_{cur} = \dfrac{T}{q B_T^{\cancel{2}}}\,\dfrac{\cancel{R}\,\cancel{B_T}}{R^{\cancel{2}}}\,\hat{z}$
- $V_{cur} = \dfrac{T}{q B_T R}\,\hat{z}$

for our geometry

In dimensionless form:

$$\boxed{\frac{v_{cur}}{v_{thi}} = \frac{T}{q B_T R\, v_{thi}}\,\hat{z}}$$

$$= \frac{(2000\,eV)(\cancel{1.6\times10^{-19}}\,J/eV)}{(\cancel{1.6\times10^{-19}}\,C)(3.41)(.66+.21\,m)(7.6\times10^{5}\,m/s)}$$

$$\boxed{\frac{V_{cur}}{v_{thi}} = 8.9\times10^{-4}}$$

→ Curvature & $\nabla B \perp B$ Drifts add!

→ only dependent on T & q