Massachusetts Institute of Technology

Subject 6.651J/8.613J/22.611J — R. Parker
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Problem Set 8

Problem 1.

In this problem, we will calculate the damping of Alfvén waves using the cold plasma two-fluid model including collisions between ions and electrons. The equilibrium plasma is considered to be homogenous and without flow.

As usual, we assume that all variables take the form $x = X\exp(ik\cdot\vec{r} - i\omega t)$ where X is a complex constant. The linearized two fluid equations can then be written in the form:

$$-i\omega m_\alpha \vec{V}_\alpha = q_\alpha \vec{E} + q_\alpha \vec{V}_\alpha \times \vec{B}_0 - m_\alpha \nu_{\alpha\alpha'}(\vec{V}_\alpha - \vec{V}_{\alpha'})$$

where n_0 is the constant equilibrium density and $\vec{B}_0$ is a uniform magnetic field which we will take in the z-direction. We will further assume that the ion charge is *e* corresponding to singly charged ions. Note from momentum conservation $m_\alpha \nu_{\alpha\alpha'} = m_{\alpha'}\nu_{\alpha'\alpha}$.

a) The solution of these equations for the ion and electron velocities is rather complicated. However, as a first step show that the ion and electron velocities satisfy the equations

$$i)\quad V_{ix} = \frac{-i\omega}{-\omega^2+\omega_{ci}^{*2}}\frac{q_i}{m_i^*}E_x + \frac{\omega_{ci}^*}{-\omega^2+\omega_{ci}^{*2}}\frac{q_i}{m_i^*}E_y - \frac{i\omega}{-\omega^2+\omega_{ci}^{*2}}\frac{m_i}{m_i^*}\nu_{ie}V_{ex} + \frac{\omega_{ci}^*}{-\omega^2+\omega_{ci}^{*2}}\frac{m_i}{m_i^*}\nu_{ie}V_{ey}$$

$$ii)\quad V_{iy} = \frac{-i\omega}{-\omega^2+\omega_{ci}^{*2}}\frac{q_i}{m_i^*}E_y - \frac{\omega_{ci}^*}{-\omega^2+\omega_{ci}^{*2}}\frac{q_i}{m_i^*}E_x - \frac{i\omega}{-\omega^2+\omega_{ci}^{*2}}\frac{m_i}{m_i^*}\nu_{ie}V_{ey} - \frac{\omega_{ci}^*}{-\omega^2+\omega_{ci}^{*2}}\frac{m_i}{m_i^*}\nu_{ie}V_{ex}$$

$$iii)\quad V_{ex} = \frac{-i\omega}{-\omega^2+\omega_{ce}^{*2}}\frac{q_e}{m_e^*}E_x + \frac{\omega_{ce}^*}{-\omega^2+\omega_{ce}^{*2}}\frac{q_e}{m_e^*}E_y - \frac{i\omega}{-\omega^2+\omega_{ce}^{*2}}\frac{m_e}{m_e^*}\nu_{ei}V_{ix} + \frac{\omega_{ce}^*}{-\omega^2+\omega_{ce}^{*2}}\frac{m_e}{m_e^*}\nu_{ei}V_{iy}$$

$$iv)\quad V_{ey} = \frac{-i\omega}{-\omega^2+\omega_{ce}^{*2}}\frac{q_e}{m_e^*}E_y - \frac{\omega_{ce}^*}{-\omega^2+\omega_{ce}^{*2}}\frac{q_e}{m_e^*}E_x - \frac{i\omega}{-\omega^2+\omega_{ce}^{*2}}\frac{m_e}{m_e^*}\nu_{ei}V_{iy} - \frac{\omega_{ce}^*}{-\omega^2+\omega_{ce}^{*2}}\frac{m_e}{m_e^*}\nu_{ei}V_{ix}$$

where $m_{i,e}^* = m_{i,e}(1 + i\dfrac{\nu_{i,e}}{\omega})$ and $\omega_{ci,ce}^* = \dfrac{m_{i,e}}{m_{i,e}^*}\omega_{ci,ce}$.

For Alfvén waves we are interested in frequencies much less than the cyclotron frequency. Accordingly, expand the denominators in these expressions using the relation

$$\frac{1}{-\omega^2+\omega_{ce,ci}^{*2}}\frac{q_{e,i}}{m_{e,i}^*}E_{x,y} \approx \frac{1}{\omega_{ce,ci}^*}(1+\frac{\omega^2}{\omega_{ce,ci}^{*2}})\frac{E_{x,y}}{B} \approx \frac{1}{\omega_{ce,ci}}(1+i(\frac{\nu_{ei,ie}}{\omega}+2\frac{\omega\nu_{ei,ie}}{\omega_{ce,ci}^2}))\frac{E_{x,y}}{B}$$

correct to first order in $\frac{\omega}{|\omega_{ce,ci}|}$ and $\frac{\nu_{ei,ie}}{\omega}$. Then further approximate the electron terms using

$$\frac{1}{\omega_{ce}}(1+i(\frac{\nu_{ei}}{\omega}+2\frac{\omega\nu_{ei}}{\omega_{ce}^2})) \approx \frac{1}{\omega_{ce}}(1+i\frac{\nu_{ei}}{\omega}).$$

and notice that

$$\frac{\omega_{ce,ci}^*}{-\omega^2+\omega_{ce,ci}^{*2}}\frac{q_{e,i}}{m_{e,i}^*}E_{x,y} \approx (1+\frac{\omega^2}{\omega_{ce,ci}^2}(1+i2\frac{\nu_{e,i}}{\omega}))\frac{E_{x,y}}{B}$$

b) Using these approximations and consistently ignoring terms of order $\omega/|\omega_{ce}|$ and ω^2/ω_{ci}^2, show that the above equations can be simplified to:

$$\text{i}')\ \ V_{ix} = \frac{-i\omega}{\omega_{ci}}(1+i(\frac{\nu_{ie}}{\omega}+2\frac{\omega\nu_{ie}}{\omega_{ci}^2}))\frac{E_x}{B}+(1+i2\frac{\omega\nu_{ie}}{\omega_{ci}^2})\frac{E_y}{B}-i\frac{\omega\nu_{ie}}{\omega_{ci}^2}V_{ex}+\frac{\nu_{ie}}{\omega_{ci}}V_{ey}$$

$$\text{ii}')\ \ V_{iy} = \frac{-i\omega}{\omega_{ci}}(1+i(\frac{\nu_{ie}}{\omega}+2\frac{\omega\nu_{ie}}{\omega_{ci}^2}))\frac{E_y}{B}-(1+i2\frac{\omega\nu_{ie}}{\omega_{ci}^2})\frac{E_x}{B}-i\frac{\omega\nu_{ie}}{\omega_{ci}^2}V_{ey}-\frac{\nu_{ie}}{\omega_{ci}}V_{ex}$$

$$\text{iii}')\ \ V_{ex} = \frac{\nu_{ei}}{\omega_{ce}}\frac{E_x}{B}+\frac{E_y}{B}+\frac{\nu_{ei}}{\omega_{ce}}V_{iy}$$

$$\text{iv}')\ \ V_{ey} = \frac{\nu_{ei}}{\omega_{ce}}\frac{E_y}{B}-\frac{E_x}{B}-\frac{\nu_{ei}}{\omega_{ce}}V_{ix}$$

to first order in the collision frequency, i.e., when only terms proportional to the collision frequency are retained. In the absence of collisions, we can see from these expressions that the ion response consists of polarization and $\vec{E}\times\vec{B}$ drift, while the electrons undergo simply the $\vec{E}\times\vec{B}$ drift.

Substituting the electron velocities into the ion equations and ignoring terms of order ν_{ei}^2/ω_{ce}^2 we get

$$\text{i}'')\ V_{ix} = \frac{-i\omega}{\omega_{ci}}(1+i2\frac{\omega\nu_{ie}}{\omega_{ci}^2})\frac{E_x}{B}+(1+i\frac{\omega\nu_{ie}}{\omega_{ci}^2})\frac{E_y}{B}$$

$$\text{ii}'')\ V_{iy} = \frac{-i\omega}{\omega_{ci}}(1+i2\frac{\omega\nu_{ie}}{\omega_{ci}^2}))\frac{E_y}{B}-(1+i\frac{\omega\nu_{ie}}{\omega_{ci}^2})\frac{E_x}{B}$$

and back substituting into the electron equations yields

$$\text{iii}'')\ V_{ex} = \frac{E_y}{B}+\frac{-i\omega}{\omega_{ci}}\frac{\nu_{ei}}{\omega_{ce}}\frac{E_y}{B}$$

$$\text{iv}'')\ V_{ey} = -\frac{E_x}{B} + \frac{i\omega}{\omega_{ci}}\frac{\nu_{ei}}{\omega_{ce}}\frac{E_x}{B}$$

In obtaining these expressions, only terms up through and including the first power of the collision frequency are retained.

Then,

$$V_{ix} - V_{ex} = \frac{-i\omega}{\omega_{ci}}(1 + i2\frac{\omega\nu_{ie}}{\omega_{ci}^2})\frac{E_x}{B}$$
$$V_{iy} - V_{ey} = \frac{-i\omega}{\omega_{ci}}(1 + i2\frac{\omega\nu_{ie}}{\omega_{ci}^2})\frac{E_y}{B}$$

and the elements of the dielectric tensor are found to be simply

$$K_\perp = \frac{\omega_{pi}^2}{\omega_{ci}^2}(1 + i\gamma), \quad K_\times = 0$$

where $\gamma = 2\frac{\omega\nu_{ie}}{\omega_{ci}^2}$.

c) Use this result to calculate the damping rate of an Alfvén wave correct to lowest order in the collision frequency.

Problem 2.

Using your favorite plotting software (Matlab, IDL, etc.) prepare a graph accurately depicting regions of propagation and cutoff in the log n^2 – $\log(\omega/\omega_{ci})$ plane for two sets of Alcator C-Mod parameters: a) $n_e = 10^{20}\ m^{-3}$, B = 5 T and b) $n_e = 4\times10^{20}\ m^{-3}$, B = 5 T. In both cases assume that the ions are deuterium. In the plots, set the abscissa range to $-2 < \log(\omega/\omega_{ci}) < 4$ and the ordinate range to $-2 < \log n^2 < 3$.