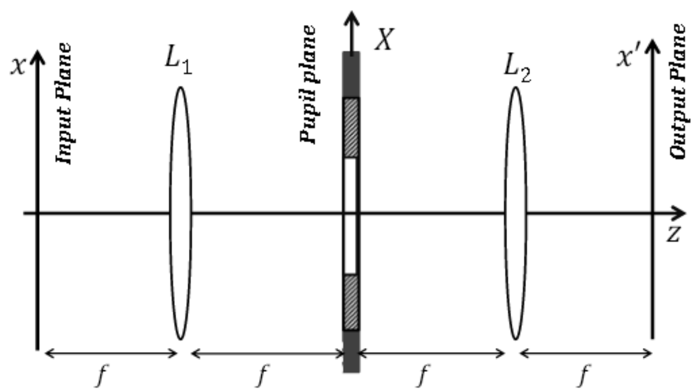
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

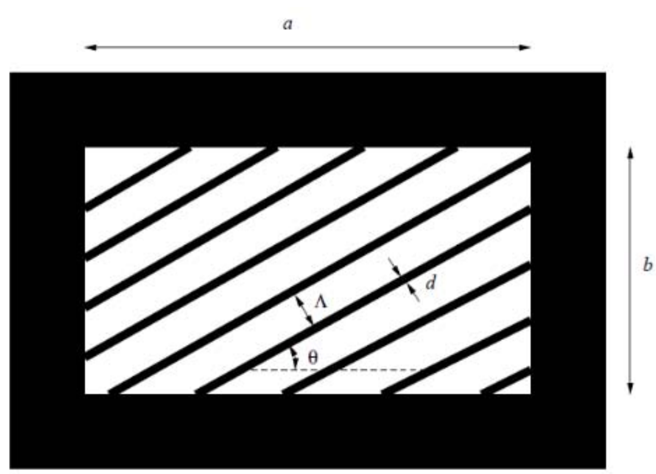
2.71/2.710 Optics **Spring ’13**

Final Exam Problem #4

(30 points) We are given a 4F imaging system consisting of two identical lenses L1, L2 with focal length f =10cm. As an experiment, you attached a tilted grating as amplitude mask at the pupil plane of the imaging system. In all questions below, the illumination is assumed to be at wavelength λ = 0.5μm. Assume spatial period Λ = 10μm, stripe size d = 2μm, tilt θ = 30◦ with respect to the aperture, and edge lengths a = 5mm, b = 3mm.

Hint: First calculate the Fourier transforms of the tilted grating and the aperture individually. Then use the convolution theorem.





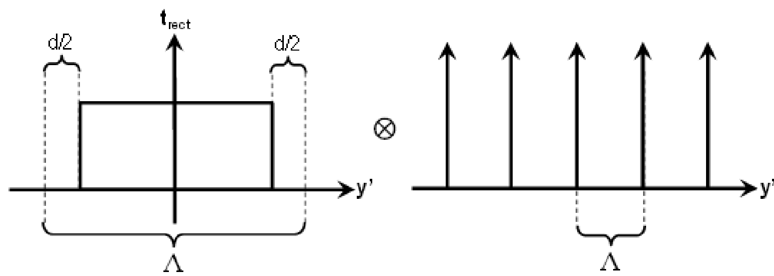
a) Let’s consider a coherent plane wave illuminating the input plane. Please calculate the output field and sketch the intensity pattern.

A coherent plane wave illuminating the input plane becomes a point on the Pupil plane, and therefore goes through the pupil mask without being modified. The output is a plane wave, and the intensity pattern is constant over x. (plot a straight line)

b) Now a coherent point source is placed at the origin of the input plane(x=0), please calculate the point spread function (PSF) and sketch the Modulation Transfer Function (MTF).

The PSF can be obtained by placing a point source at the origin and calculating the field at the output plane. The point source is transformed by the first lens into a plane wave illuminating the pupil mask. This field can be mathematically represented as a tilted grating multiplied with a large rectangular aperture.

First consider the grating; it can be written as a convolution of a comb function with a rectangle:



$$t_{rect}(x', y') = \mathrm{rect}\left(\frac{y'}{\Lambda - d}\right) \otimes \mathrm{comb}\left(\frac{y'}{\Lambda}\right) \qquad [3\text{-}1]$$

where x' is the coordinate along the tilted lines and y' is perpendicular to them. The rotation matrix can be used to relate the tilted coordinate system with the native xy-plane:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad [3\text{-}2]$$

The outer rectangular aperture can be written as:

$$t_{aperture}(x, y) = \mathrm{rect}\left(\frac{x}{a}\right)\ \mathrm{rect}\left(\frac{y}{b}\right) \qquad [3\text{-}3]$$

The object transmission function is the product of equations [3-1] and [3-2]:

$$\begin{aligned} t(x, y) &= t_{aperture}(x, y) t_{rect}(x, y) \\ &= \mathrm{rect}\left(\frac{x}{a}\right)\ \mathrm{rect}\left(\frac{y}{b}\right)\left[\mathrm{rect}\left(\frac{x\sin\theta + y\cos\theta}{\Lambda - d}\right) \otimes \mathrm{comb}\left(\frac{x\sin\theta + y\cos\theta}{\Lambda}\right)\right] \end{aligned} \qquad [3\text{-}4]$$

The second lens Fourier transforms this pupil plane into the output plane. The Fourier transform of the pupil mask can be calculated as the following.

Taking the Fourier Transform of Equation [3-4] would provide the answer, however, it is not straightforward how to take the Fourier Transform of a single rect that depends on two different coordinates. Therefore, it is more straightforward to keep the rect and comb functions in terms of the tilted coordinate system:

$$t(x,y;x',y') = \mathrm{rect}\left(\frac{x}{a}\right)\ \mathrm{rect}\left(\frac{y}{b}\right)\left[\mathrm{rect}\left(\frac{y'}{\Lambda-d}\right)\otimes\mathrm{comb}\left(\frac{y'}{\Lambda}\right)\right] \qquad [3\text{-}5]$$

Taking the Fourier Transform of Equation [3-5] and using the convolution theorem:

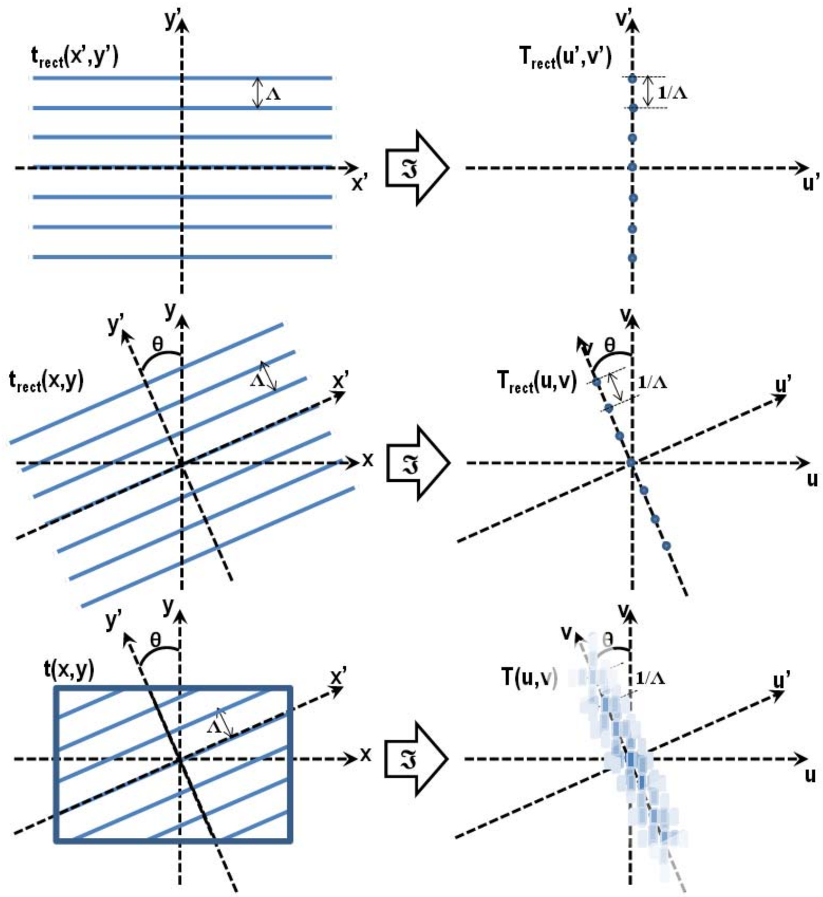
$$g_{out} = \Im\{t\} \propto \mathrm{sinc}(au)\mathrm{sinc}(bv)\otimes\left[\mathrm{sinc}\big((\Lambda-d)v'\big)\mathrm{comb}(\Lambda v')\right] \qquad [3\text{-}6]$$

A tilted spatial coordinate system corresponds to a tilted frequency-space with the same rotation. Therefore:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \qquad [3\text{-}7]$$

Therefore the output field is:

$$\boxed{g_{out} = \mathrm{sinc}\big((\Lambda-d)(u\mathrm{sin}\theta+v\mathrm{cos}\theta)\big)\mathrm{sinc}(au)\mathrm{sinc}(bv)\otimes\mathrm{comb}\big(\Lambda(u\mathrm{sin}\theta+v\mathrm{cos}\theta)\big) \qquad [3\text{-}8]}$$



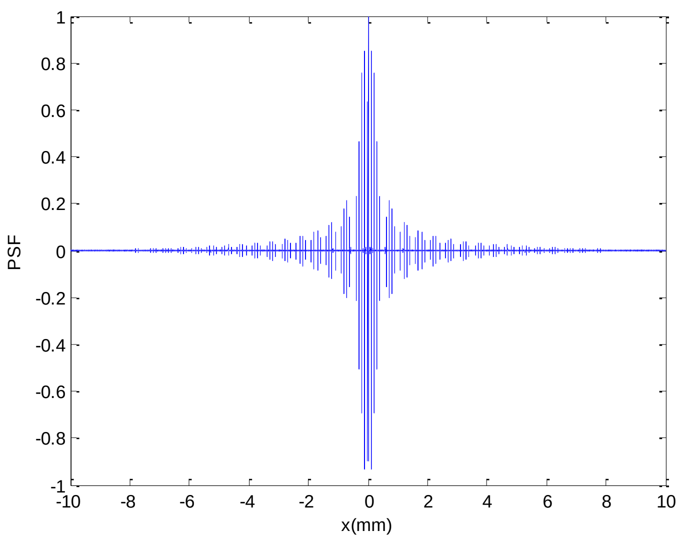
(Fourier transform solution is adapted from Pset6-P3 of 2012. solution credit: Matt Klug)

Now, the PSF can be obtained by substituting $u = x'/\lambda f, v = y'/\lambda f$ into the Fourier transform result [3-8]:

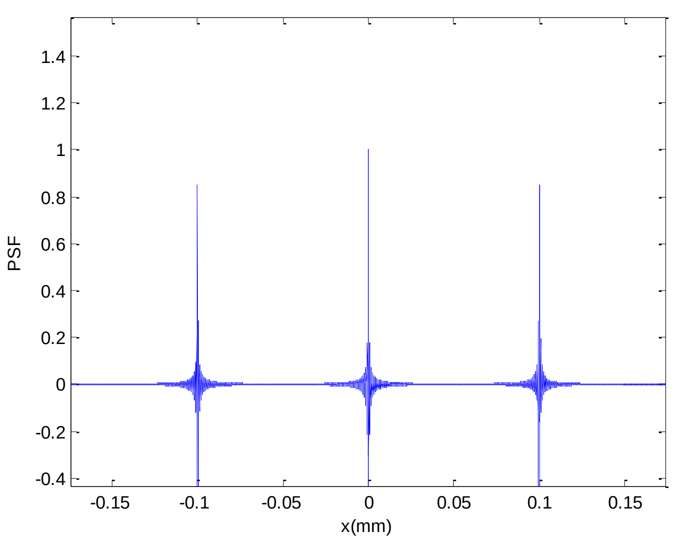
$$h(x', y') = \mathrm{sinc}(\mathrm{a}\frac{x'}{\lambda f})\mathrm{sinc}(\mathrm{b}\frac{\mathrm{y'}}{\lambda f}) \otimes \left[\mathrm{sinc}\left((\Lambda - d)(\frac{x'}{\lambda f}\sin\theta + \frac{\mathrm{y'}}{\lambda f}\cos\theta) \right) comb\left(\Lambda(\frac{x'}{\lambda f}\sin\theta + \frac{\mathrm{y'}}{\lambda f}\cos\theta) \right) \right].$$

The plot of the PSF is given below.

Overall :



Zoomed in:



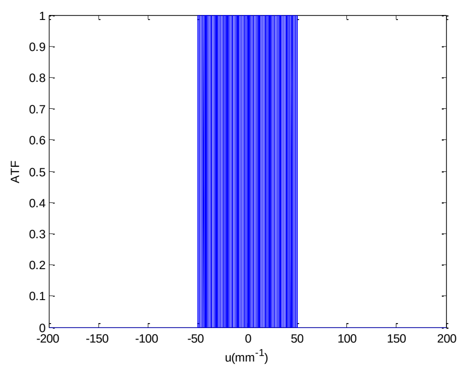
The MTF is the modulus of the OTF. Recall that the ATF is the Fourier transform of the PSF, and the OTF is the Fourier transform of the $\text{iPSF} = |h(x,y)|^2$. The iPSF is obtained from squaring the PSF, and OTF is obtained from calculating the autocorrelation of the ATF. $\text{OTF}=\text{ATF}*\text{ATF}$, where * represents autocorrelation. The definition of the autocorrelation function is given below. This is very similar to a convolution.

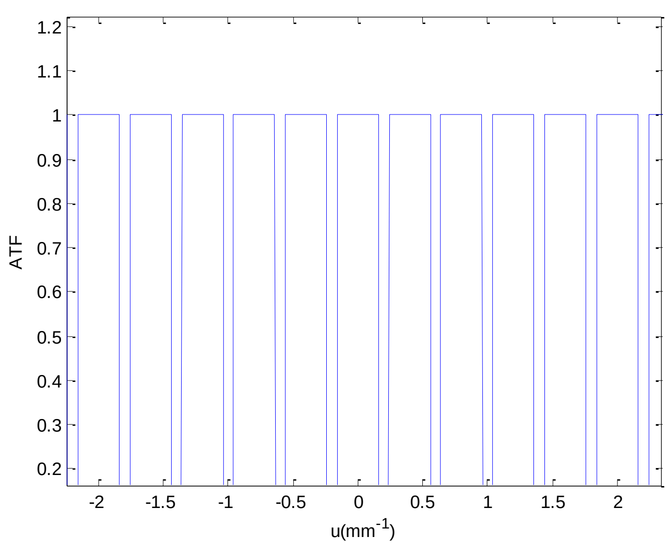
$$f * f\ (\eta) = \int f(x) f(x-\eta) dx$$

(Convolution is flip-shift-multiply-integrate, while autocorrelation is simply shift-multiply-integrate, without changing the sign of the argument of the second function)

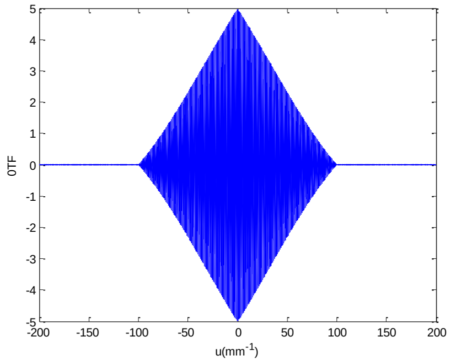
The ATF, OTF, and MTF are plotted as a function of $u = \dfrac{X}{\lambda f}$.

The ATF of the system is a scaled version of the amplitude mask itself. Plotting with respect to the given parameters, we get: (Overall / Zoomed in)

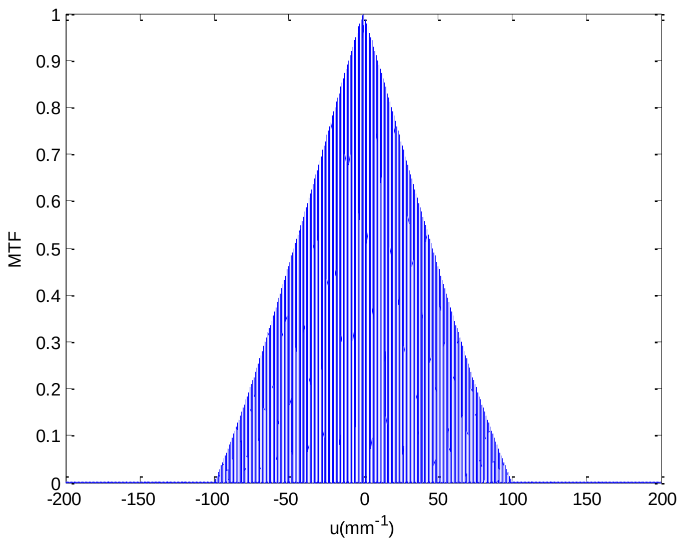


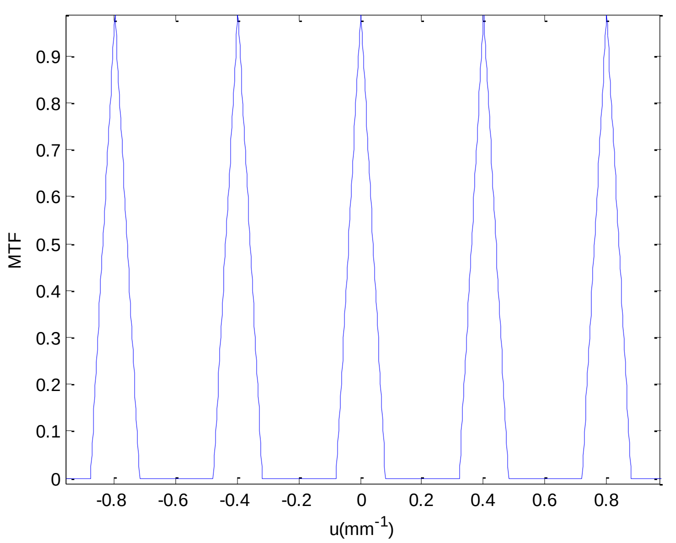


Re(OTF):



The MTF of the system is (Overall/ Zoomed in):



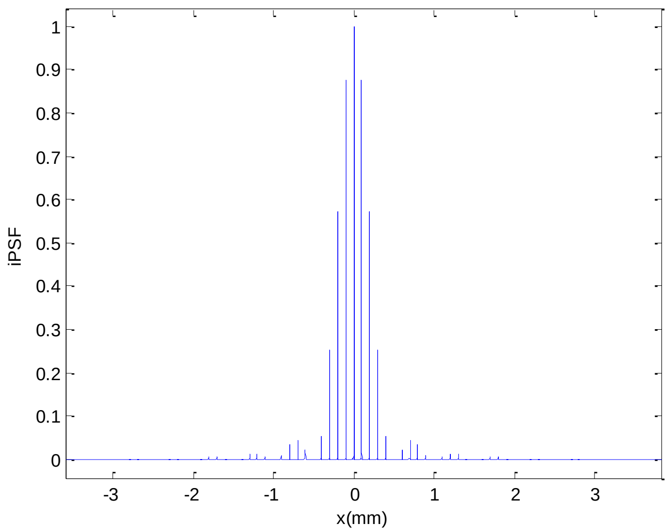


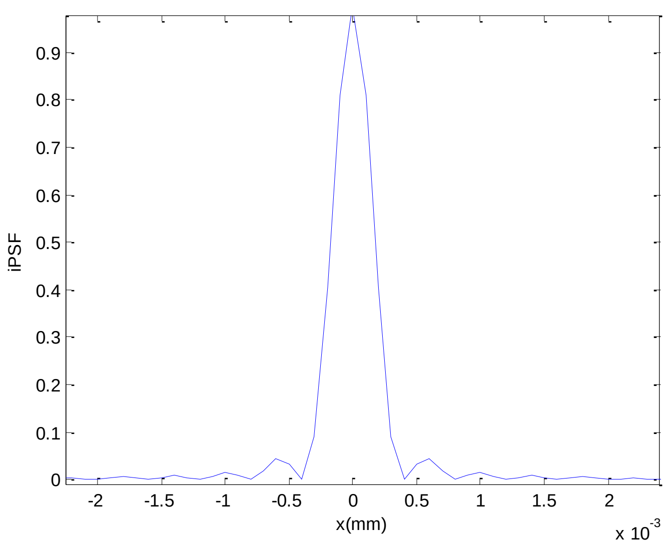
c) (2.710 only) When the illumination is spatially incoherent, please calculate and plot the intensity point spread function (iPSF).

The iPSF can be calculated as:

$$|h(x',y')|^2 = \left[\mathrm{sinc}(a\frac{x'}{\lambda f})\mathrm{sinc}(b\frac{y'}{\lambda f}) \otimes \left[\mathrm{sinc}\left((\Lambda - d)(\frac{x'}{\lambda f}\sin\theta + \frac{y'}{\lambda f}\cos\theta) \right) comb\left(\Lambda(\frac{x'}{\lambda f}\sin\theta + \frac{y'}{\lambda f}\cos\theta) \right) \right] \right]^2.$$

The plot of the iPSF is (Overall/Zoomed in center lobe):





2.71 / 2.710 Optics
Spring 2014