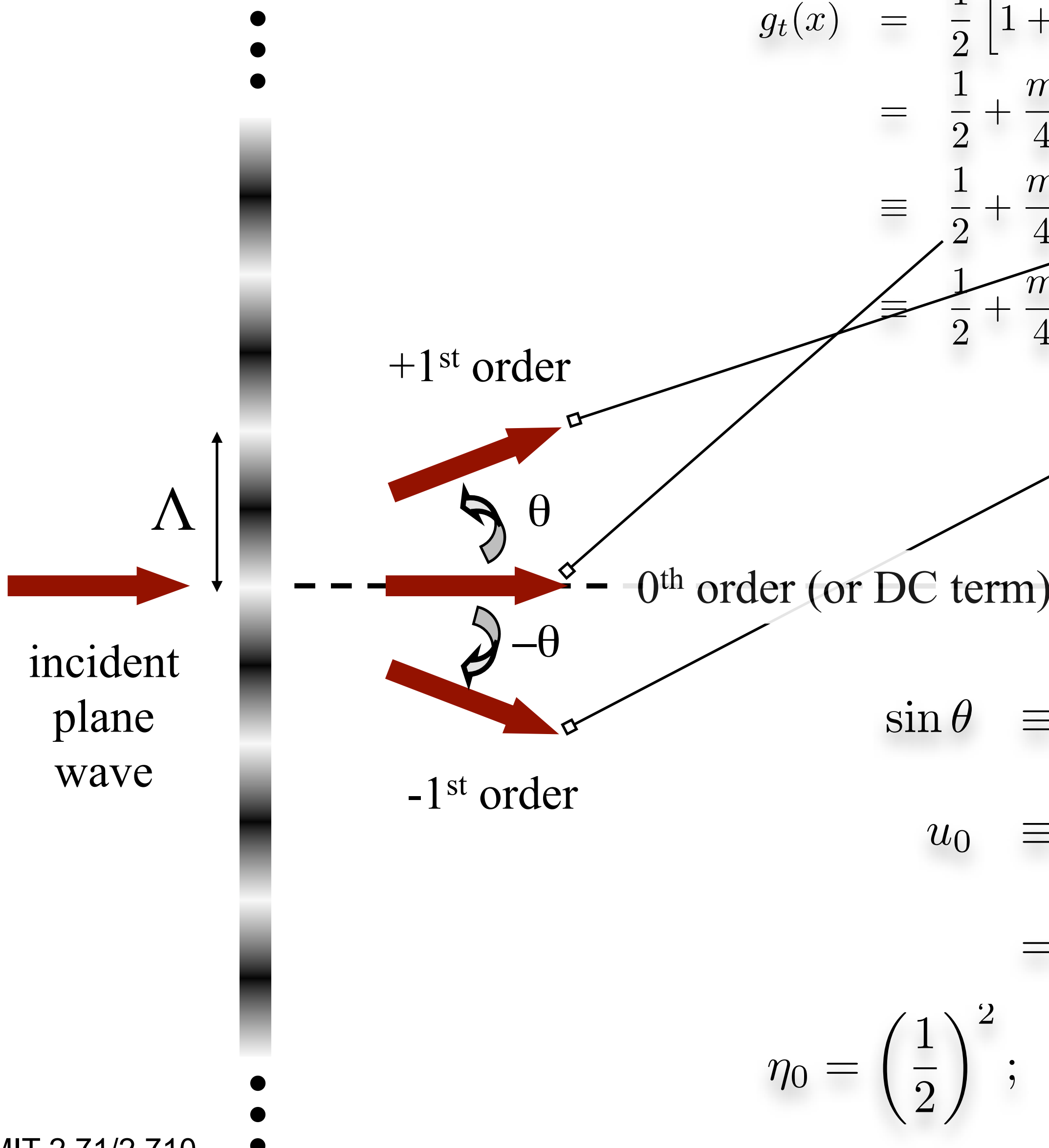
Sinusoidal amplitude grating

$$
\begin{aligned}
g_t(x) &= \frac{1}{2}\left[1 + m\cos\left(2\pi\frac{x}{\Lambda} + \phi\right)\right] \\
&= \frac{1}{2} + \frac{m}{4}\exp\left\{i2\pi\frac{x}{\Lambda}\right\} + \frac{m}{4}\exp\left\{-i2\pi\frac{x}{\Lambda}\right\} \\
&\equiv \frac{1}{2} + \frac{m}{4}\exp\left\{i2\pi\frac{\sin\theta}{\lambda}x\right\} + \frac{m}{4}\exp\left\{-i2\pi\frac{\sin\theta}{\lambda}x\right\} \\
&\equiv \frac{1}{2} + \frac{m}{4}\exp\left\{i2\pi u_0 x\right\} + \frac{m}{4}\exp\left\{-i2\pi u_0 x\right\}.
\end{aligned}
$$

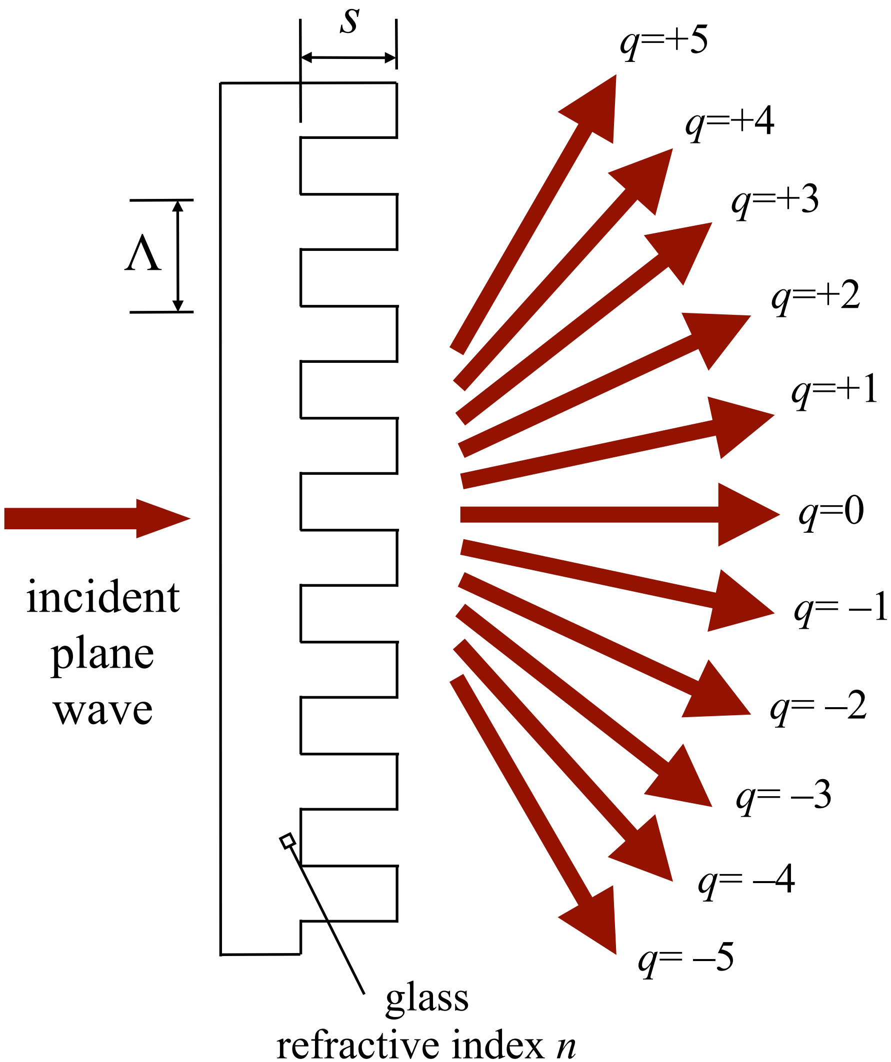


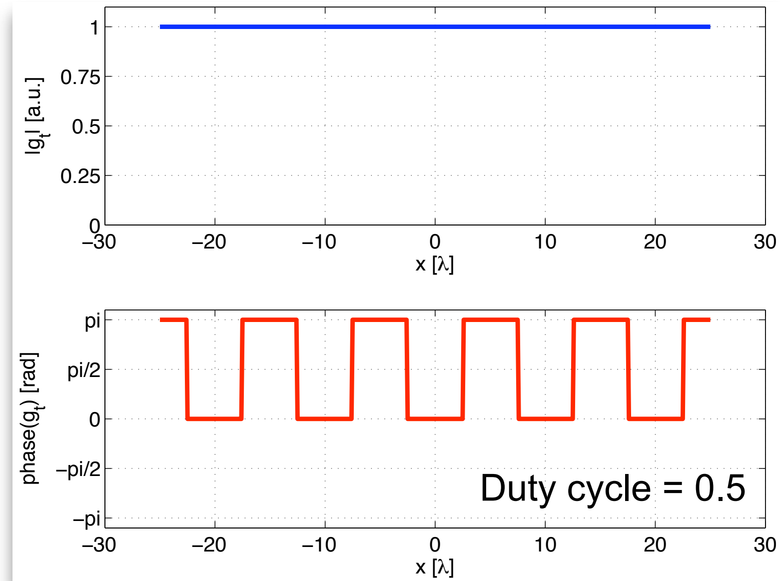
$$\sin\theta \equiv \frac{\lambda}{\Lambda};$$ diffraction angle

$$u_0 \equiv \frac{1}{\Lambda} = \frac{\sin\theta}{\lambda}.$$ spatial frequency

$$\eta_0 = \left(\frac{1}{2}\right)^2; \quad \eta_{\pm 1} = \left(\frac{m}{4}\right)^2$$ diffraction efficiencies

Example: binary phase grating





$$g_0(x) = \begin{cases} 1, & 0 \le |x| \le \Lambda/4 \\ -1, & \Lambda/4 < |x| \le \Lambda/2 \end{cases}$$

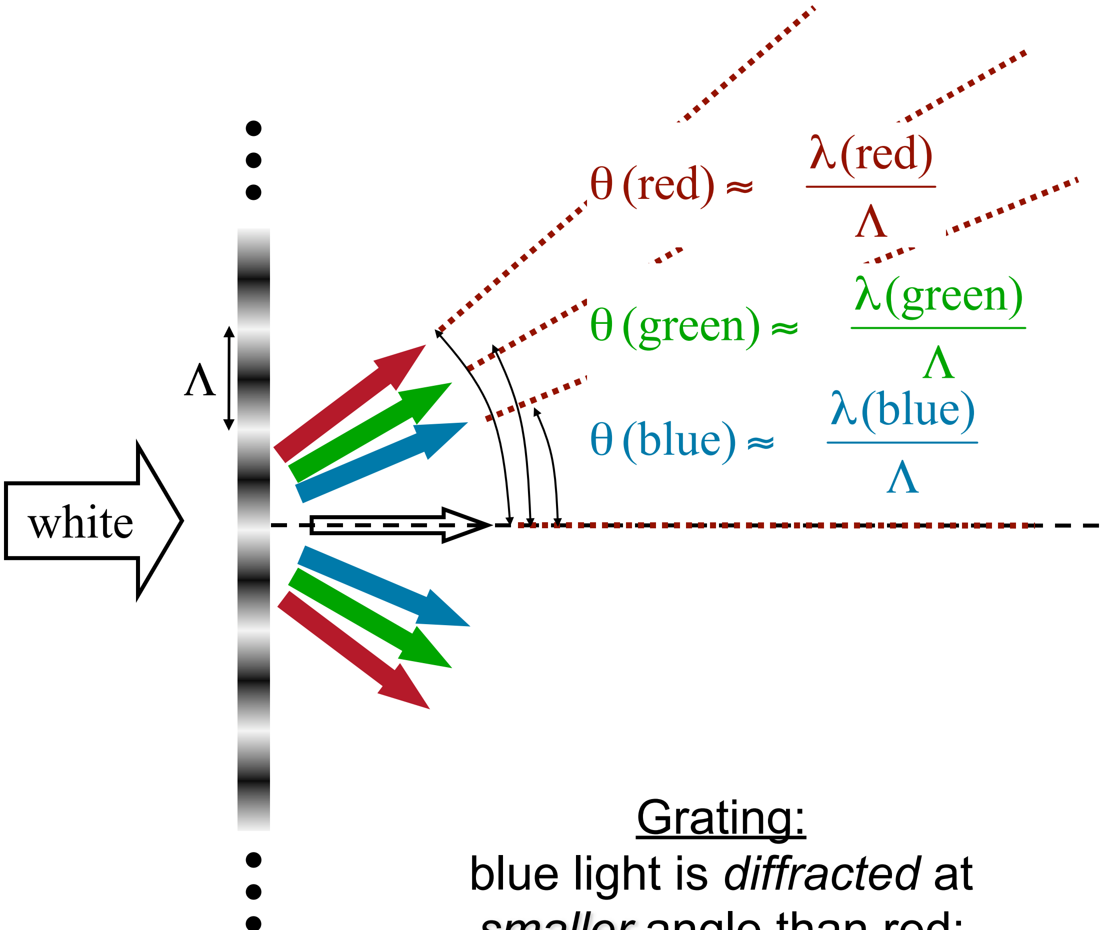
$$c_q = \frac{1}{\Lambda}\int_{-\Lambda/2}^{\Lambda/2} g_0(x)\exp\left\{i2\pi q\frac{x}{\Lambda}\right\}\mathrm{d}x.$$

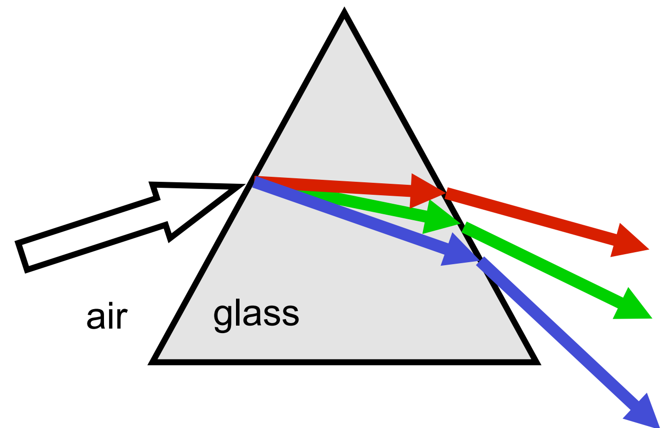
$$c_q = \mathrm{sinc}\left(\frac{q}{2}\right) \quad \text{where} \quad \mathrm{sinc}\left(\xi\right) \equiv \frac{\sin(\pi\xi)}{(\pi\xi)}.$$

$$\eta_{\pm q} = \left(\frac{2}{\pi q}\right)^2 \quad \text{for} \quad q \text{ odd.}$$

$$\eta_{\pm 1} = \left(\frac{2}{\pi}\right)^2 \approx 40.53\%.$$

Grating dispersion





<u>Grating:</u>
blue light is *diffracted* at *smaller* angle than red:

anomalous dispersion

<u>Prism:</u>
blue light is *refracted* at *larger* angle than red:

normal dispersion

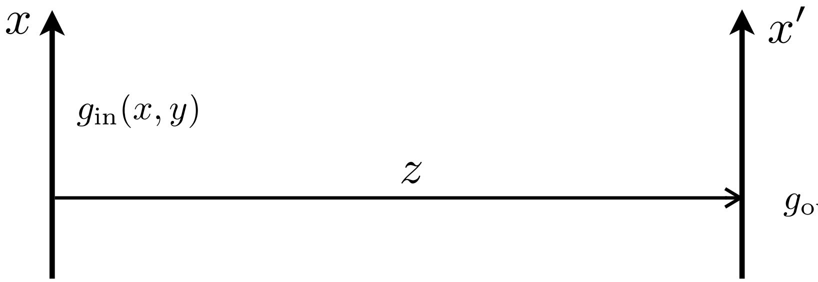
Today

- Fraunhofer diffraction
- Fourier transforms: maths
- Fraunhofer patterns of typical apertures
- Fresnel propagation: Fourier systems description
  - impulse response and transfer function
  - example: Talbot effect

Next week

- Fourier transforming properties of lenses
- Spatial frequencies and their interpretation
- Spatial filtering

Fraunhofer diffraction



Fresnel (free space) propagation may be expressed as a convolution integral

$$g_{\text{out}}(x',y') = g_{\text{in}}(x,y) \star \left(\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\} \right)$$

$$g_{\text{out}}(x',y';z) = \frac{1}{i\lambda z} \exp\left\{ i2\pi\frac{z}{\lambda} \right\} \iint g_{\text{in}}(x,y) \exp\left\{ i\pi \frac{(x'-x)^2+(y'-y)^2}{\lambda z} \right\} \mathrm{d}x\mathrm{d}y$$

If the propagation distance becomes very large $z \to \infty$, we can approximate the free–space (Fresnel) propagation integral as

$$\begin{aligned} g_{\text{out}}(x',y';z) &= \frac{1}{i\lambda z} \exp\left\{ i2\pi\frac{z}{\lambda} \right\} \iint g_{\text{in}}(x,y) \exp\left\{ i\pi \frac{x'^2+x^2-2xx'+y'^2+y^2-2yy'}{\lambda z} \right\} \mathrm{d}x\mathrm{d}y \\ &\approx \exp\left\{ i2\pi\frac{z}{\lambda} + i\pi\frac{x'^2+y'^2}{\lambda z} \right\} \iint g_{\text{in}}(x,y) \exp\left\{ -i2\pi \frac{xx'+2yy'}{\lambda z} \right\} \mathrm{d}x\mathrm{d}y \end{aligned}$$

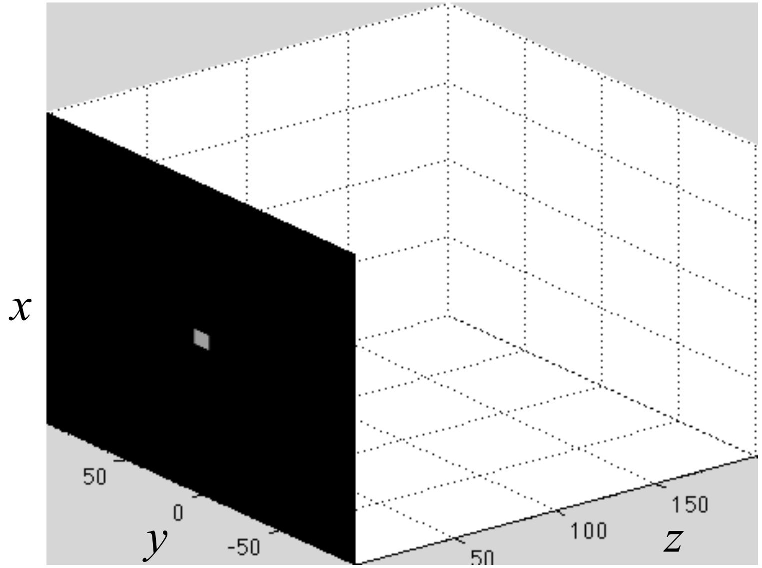
We set $u \equiv \dfrac{x'}{\lambda z} \quad v \equiv \dfrac{y'}{\lambda z}$

and rewrite the approximated Fresnel integral as

$$g_{\text{out}}(x',y';z) \approx \exp\left\{ i2\pi\frac{z}{\lambda} + i\pi\frac{x'^2+y'^2}{\lambda z} \right\} \iint g_{\text{in}}(x,y) \exp\left\{ -i2\pi\left(ux+vy\right) \right\} \mathrm{d}x\mathrm{d}y,$$

which we recognize as proportional to the Fourier transform $G_{\text{in}}(u,v)$.

Example: rectangular aperture

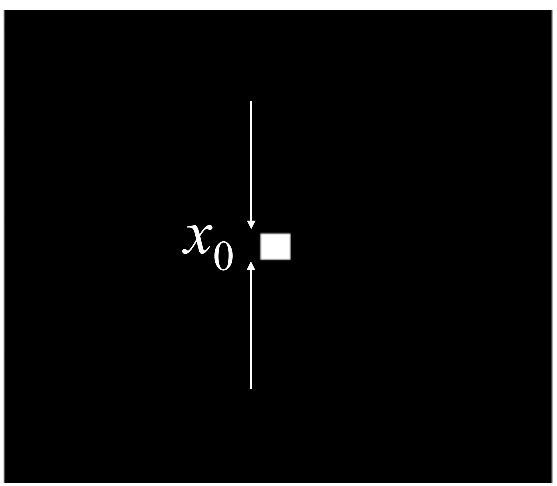


$$g_{\text{in}}(x,y) = \text{rect}\left(\frac{x}{x_0}\right)\text{rect}\left(\frac{y}{y_0}\right)$$

$$G_{\text{in}}(u,v) = x_0 y_0 \text{sinc}\left(x_0 u\right)\text{sinc}\left(y_0 v\right)$$

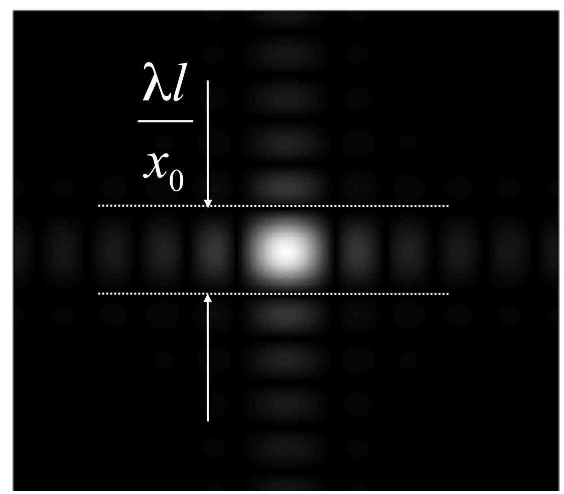
$$g_{\text{out}}(x',y';z\to\infty) \propto \text{sinc}\left(\frac{x_0 x'}{\lambda z}\right)\text{sinc}\left(\frac{y_0 y'}{\lambda z}\right).$$

sinc pattern



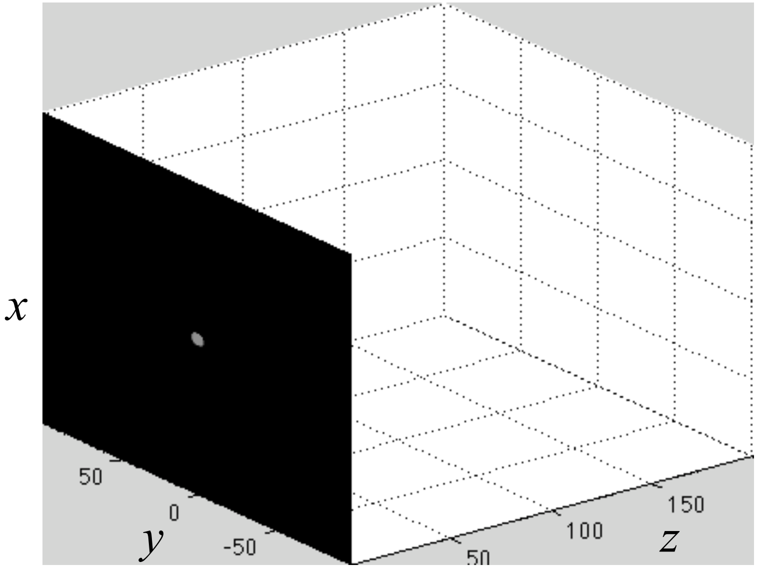
input field

free space propagation by $l \to \infty$



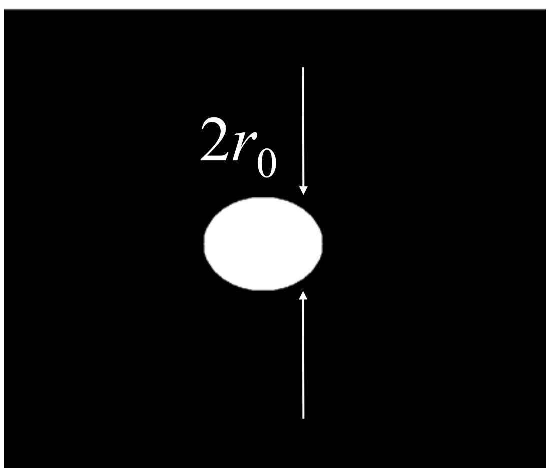
far field

Example: circular aperture



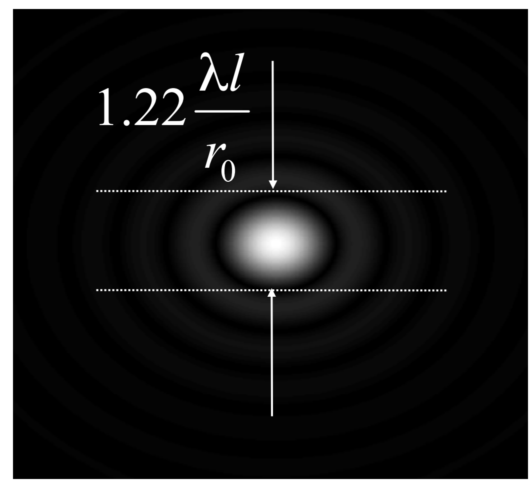
$$
\begin{aligned}
g_{\text{in}}(x,y) &= \operatorname{circ}\left(\frac{\sqrt{x^2+y^2}}{r_0}\right)\\
G_{\text{in}}(u,v) &= r_0^2 \operatorname{jinc}\left(r_0\sqrt{u^2+v^2}\right)\\
&\equiv r_0 \frac{\mathrm{J}_1\left(2\pi\sqrt{u^2+v^2}\right)}{\sqrt{u^2+v^2}}\\
g_{\text{out}}(x',y';z\to\infty) &\propto \operatorname{jinc}\left(\frac{2\pi r_0\sqrt{x'^2+y'^2}}{\lambda z}\right).
\end{aligned}
$$

Airy pattern



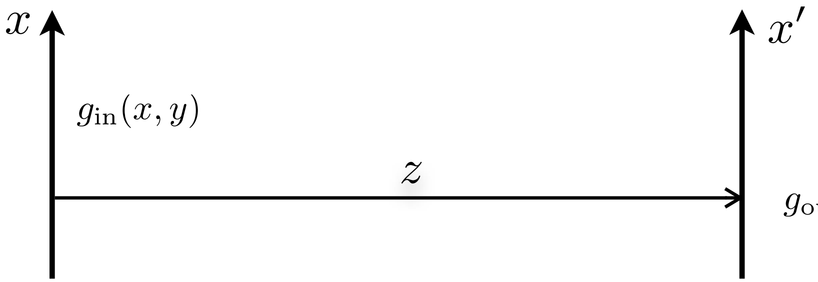
input field

free space propagation by $l\to\infty$



far field

How far along z does the Fraunhofer pattern appear?



Fresnel (free space) propagation may be expressed as a convolution integral

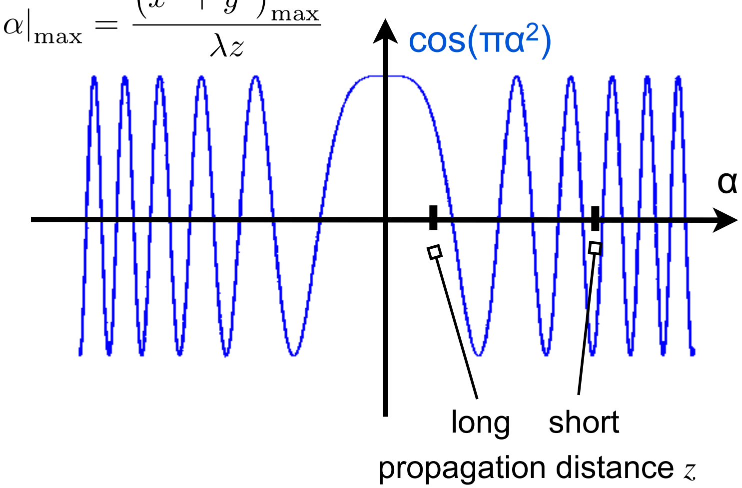
$$g_{\rm out}(x',y') = g_{\rm in}(x,y) \star \left(\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\} \right)$$

$$g_{\rm out}(x',y';z) = \frac{1}{i\lambda z} \exp\left\{ i2\pi\frac{z}{\lambda} \right\} \iint g_{\rm in}(x,y) \exp\left\{ i\pi \frac{(x'-x)^2+(y'-y)^2}{\lambda z} \right\} \mathrm{d}x\mathrm{d}y$$

$$g_{\rm out}(x',y';z) = \frac{1}{i\lambda z} \exp\left\{ i2\pi\frac{z}{\lambda} + i\pi\frac{x'^2+y'^2}{\lambda z} \right\} \iint g_{\rm in}(x,y) \exp\left\{ -i2\pi \frac{xx'+2yy'}{\lambda z} \right\} \boxed{\exp\left\{ i\pi\frac{x^2+y^2}{\lambda z} \right\}} \mathrm{d}x\mathrm{d}y$$

$$\begin{aligned} \exp\left\{ i\pi\frac{x^2+y^2}{\lambda z} \right\} &= \cos\left\{ \pi\frac{x^2+y^2}{\lambda z} \right\} + i\sin\left\{ \pi\frac{x^2+y^2}{\lambda z} \right\} \\ &\approx 1 \quad \text{if} \ \frac{\left(x^2+y^2\right)_{\max}}{\lambda z} \ll 1 \\ &\Leftrightarrow z \gg \frac{\left(x^2+y^2\right)_{\max}}{\lambda} \end{aligned}$$

$$|\alpha|_{\max} = \frac{\left(x^2+y^2\right)_{\max}}{\lambda z}$$



For example, if $(x^2+y^2)_{max}=(4\lambda)^2$, then $z \gg 16\lambda$ to enter the Fraunhofer regime;
if $(x^2+y^2)_{max}=(1000\lambda)^2$, then $z \gg 10^6\lambda$;
in practice, the Fraunhofer intensity pattern is recognizable at smaller z than these predictions (but the correct Fraunhofer phase takes longer to form)

Fourier transforms

- One dimensional
  - Fourier transform

$$G(\nu) = \int_{-\infty}^{+\infty} g(t) \exp\left\{ -i2\pi\nu t \right\} \mathrm{d}t.$$

  - Fourier integral

$$g(t) = \int_{-\infty}^{+\infty} G(\nu) \exp\left\{ i2\pi\nu t \right\} \mathrm{d}\nu.$$

- Two dimensional
  - Fourier transform

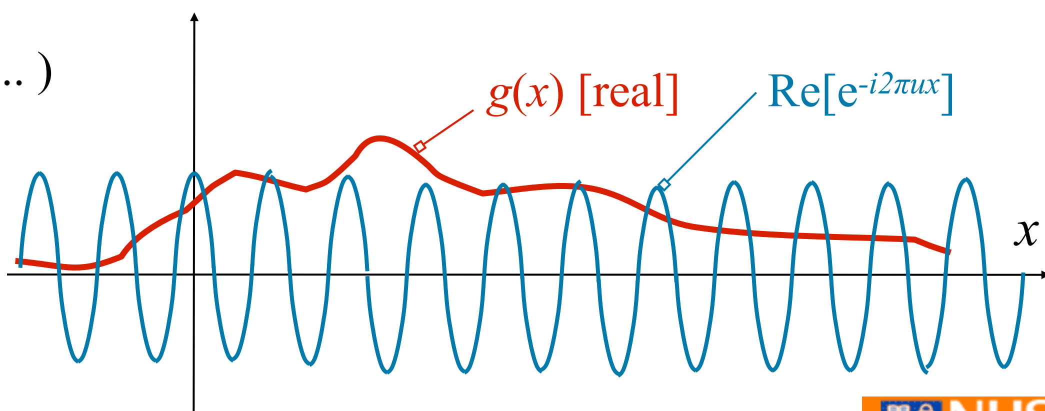
$$G(u,v) = \iint_{-\infty}^{+\infty} g(x,y) \exp\left\{ -i2\pi(ux+vy) \right\} \mathrm{d}x\mathrm{d}y.$$

  - Fourier integral

$$g(x,y) = \iint_{-\infty}^{+\infty} G(u,v) \exp\left\{ i2\pi(ux+vy) \right\} \mathrm{d}u\mathrm{d}v.$$

(1D so we can draw it easily ...)

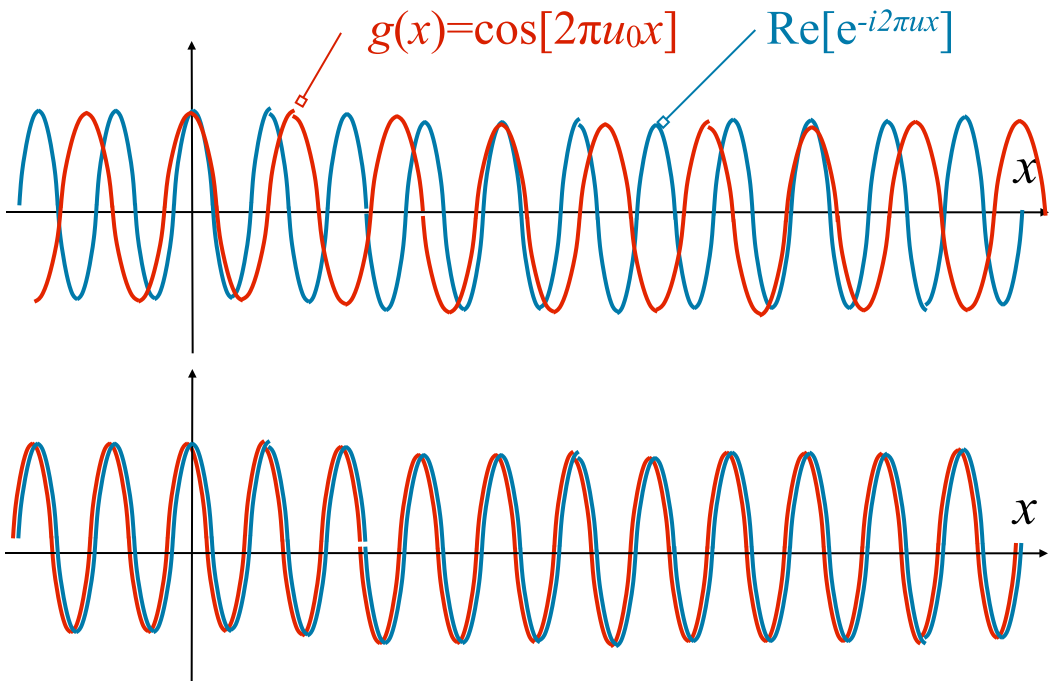
$\mathrm{Re}[G(u)] = \int$



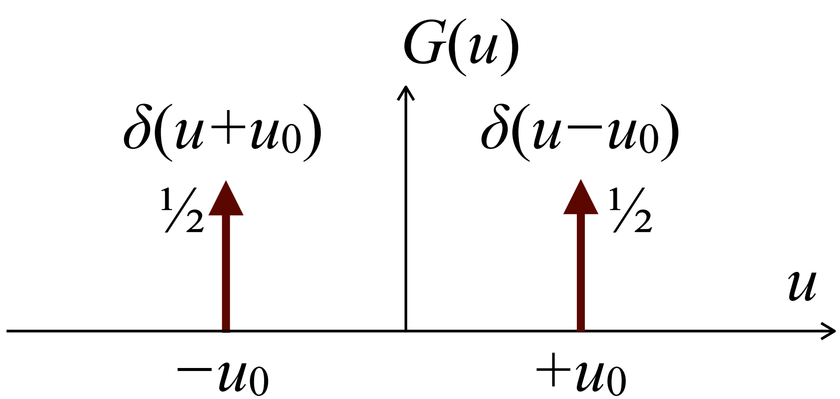
$\mathrm{d}x$

Frequency representation

$\mathrm{Re}[G(u)] = \int \ldots \, dx = 0$, if $u_0 \neq u$



$\mathrm{Re}[G(u)] = \int \ldots \, dx = \infty$, if $u_0 = u$

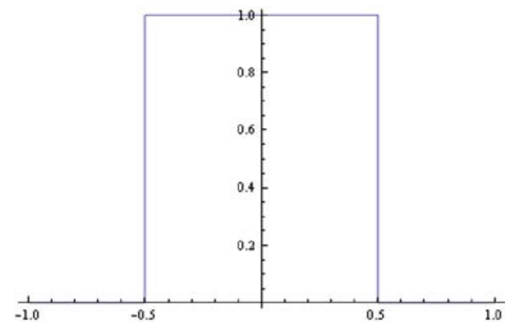


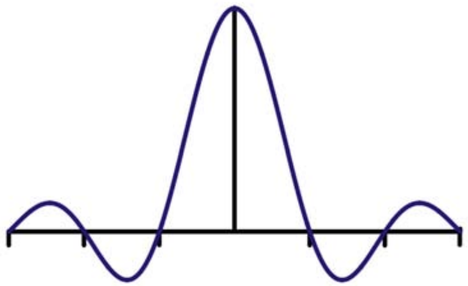
$$G(u) = \tfrac{1}{2}\,\delta(u+u_0) + \tfrac{1}{2}\,\delta(u-u_0)$$

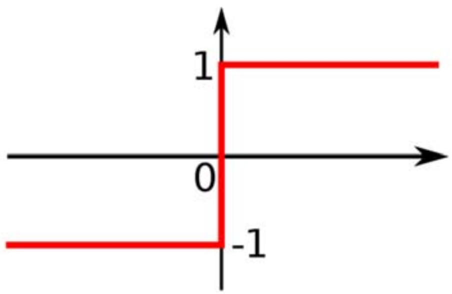
The negative frequency is physically meaningless,
but necessary for mathematical rigor;
it is the price to pay for the convenience of using
complex exponentials in the phasor representation

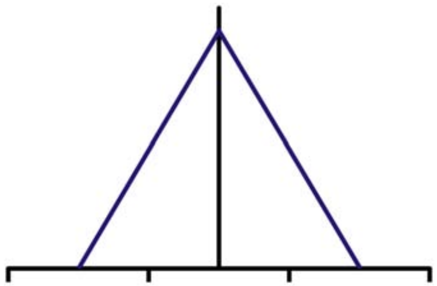
Commonly used functions in wave Optics

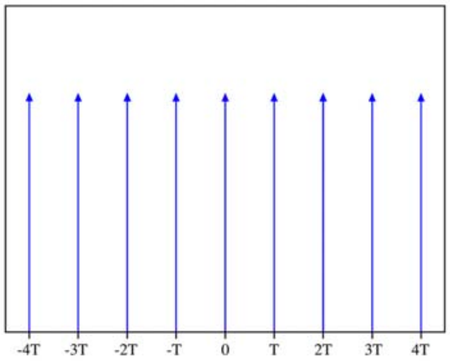
Text removed due to copyright restrictions. Please see p. 12 in Goodman, Joseph W. *Introduction to Fourier Optics*. Englewood, CO: Roberts & Co., 2004. ISBN: 9780974707723.

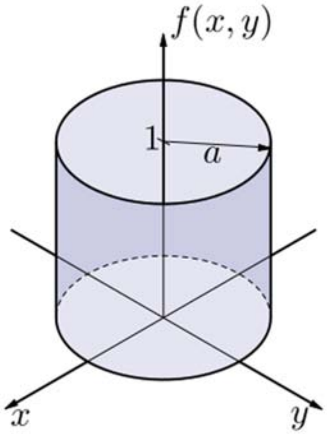












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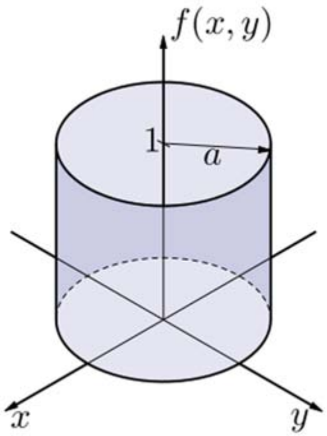
Goodman, *Introduction to Fourier Optics* (3rd ed.) pp. 12-14

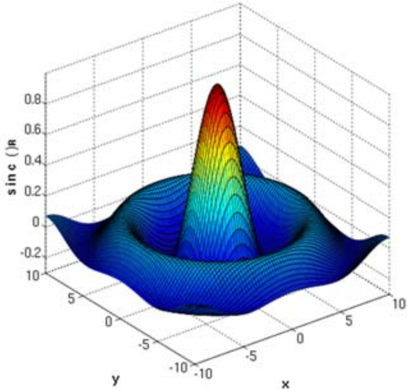
Fourier transform pairs

Table removed due to copyright restrictions. Please see Table 2.1 in
Goodman, Joseph W. *Introduction to Fourier Optics*. Englewood, CO: Roberts & Co., 2004.
ISBN: 9780974707723.

Functions with radial symmetry



jinc(ρ)≡



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$$r = \sqrt{x^2 + y^2}$$
$$\rho = \sqrt{f_X^2 + f_Y^2}$$

Goodman, *Introduction to Fourier Optics* (3rd ed.) p. 14

Fourier transform properties

A general discussion of the properties of Fourier transforms may also be found here
http://en.wikipedia.org/wiki/Fourier_transform#Properties_of_the_Fourier_transform.

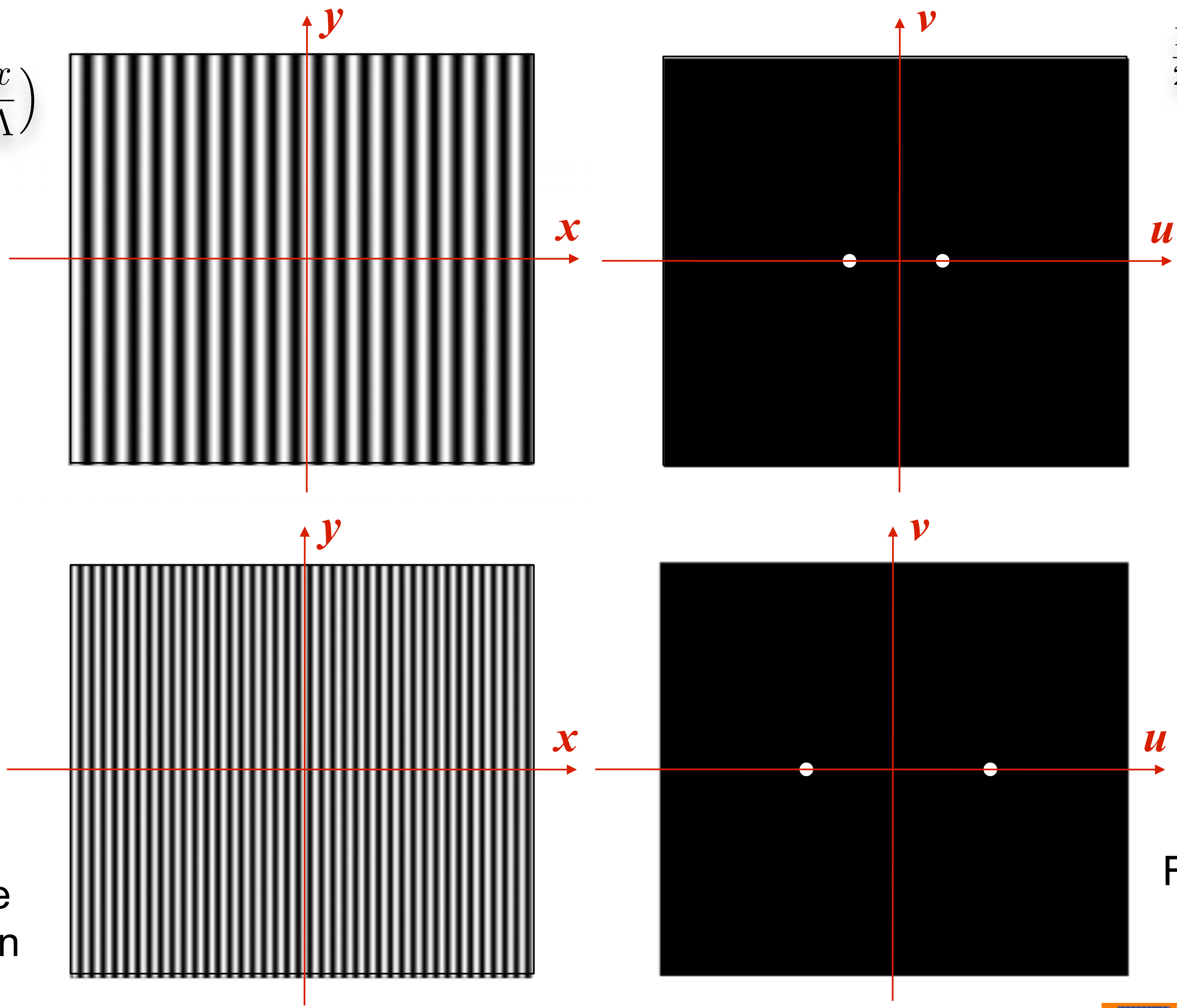
IMPORTANT! A note on notation: Goodman uses (f_X, f_Y) to denote spatial frequencies along the (x,y) dimensions, respectively.
In these notes, we will sometimes use (u,v) instead.

Goodman, *Introduction to Fourier Optics* (3rd ed.) pp. 8-9

The spatial frequency domain: vertical grating

$$\cos\left(2\pi\frac{x}{\Lambda}\right)$$

$$\frac{1}{2}\delta\left(u+\frac{1}{\Lambda}\right)\delta\left(v\right)+$$
$$\frac{1}{2}\delta\left(u-\frac{1}{\Lambda}\right)\delta\left(v\right)$$



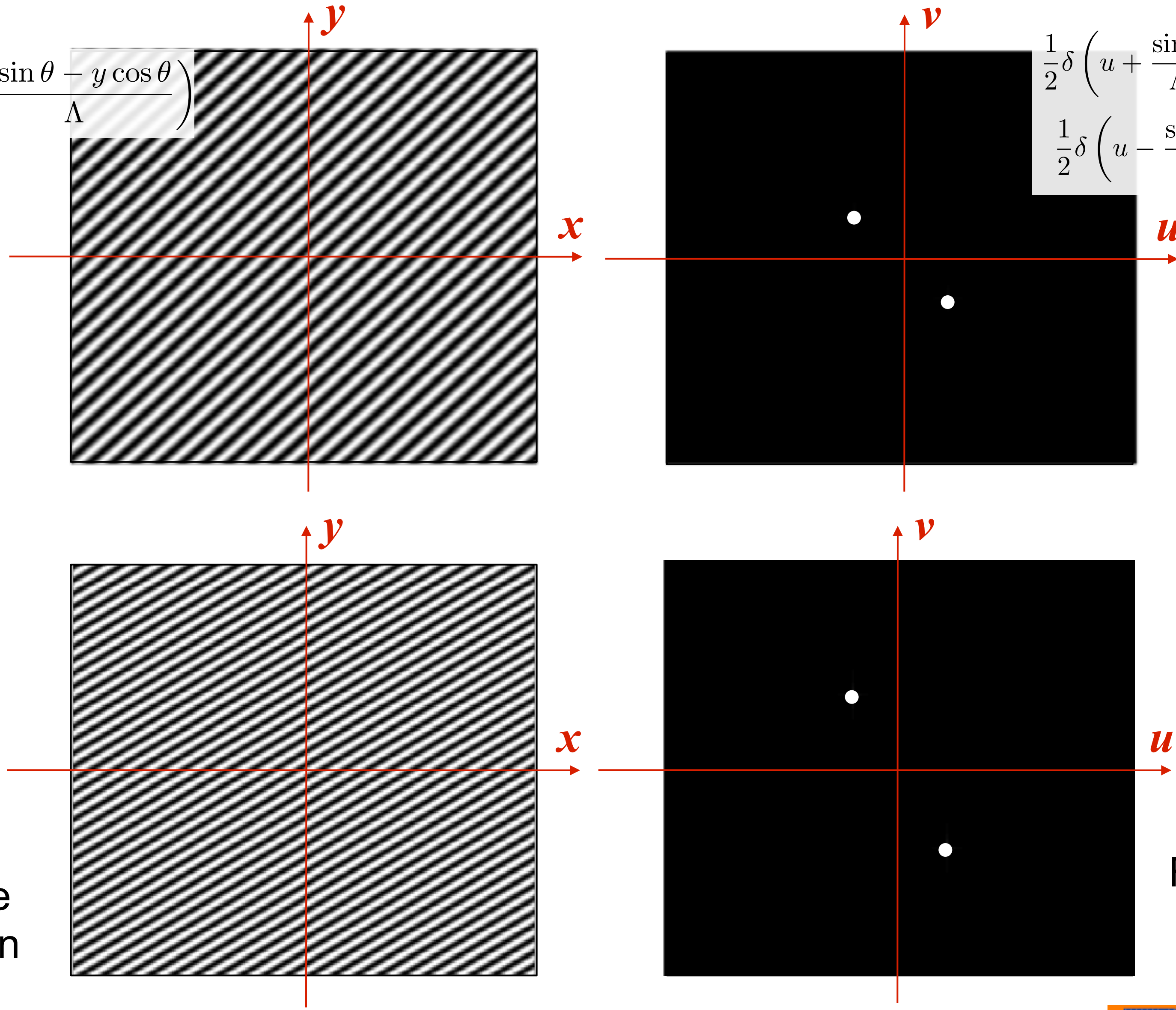
Space domain

Frequency (Fourier) domain

The spatial frequency domain: tilted grating

$$\cos\left(2\pi\frac{x\sin\theta - y\cos\theta}{\Lambda}\right)$$

$$\frac{1}{2}\delta\left(u+\frac{\sin\theta}{\Lambda}\right)\delta\left(v-\frac{\cos\theta}{\Lambda}\right)+\frac{1}{2}\delta\left(u-\frac{\sin\theta}{\Lambda}\right)\delta\left(v+\frac{\cos\theta}{\Lambda}\right)$$



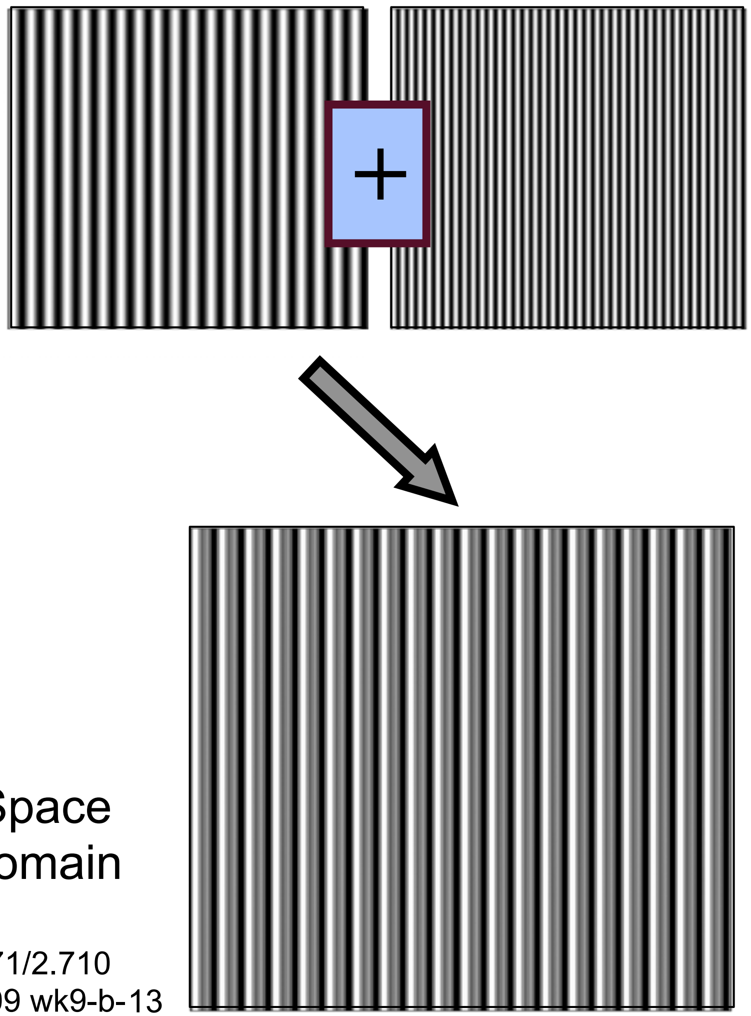
Space domain

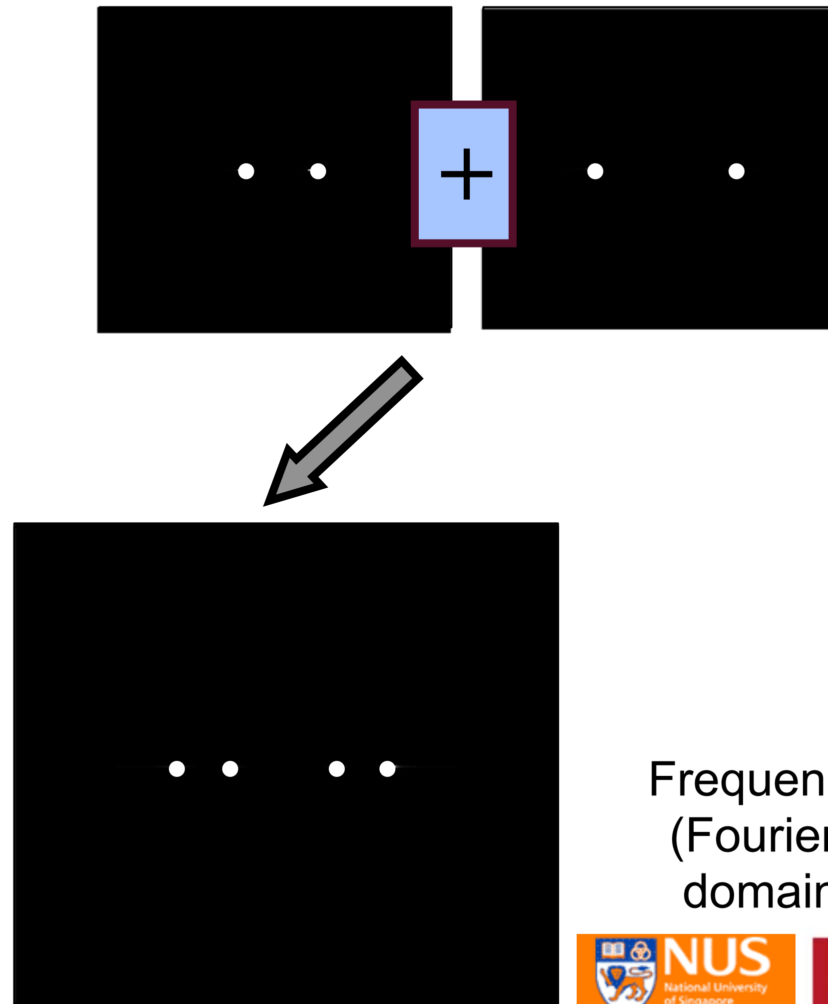
Frequency (Fourier) domain

Superposition: two gratings

$$a_1 \cos\left(2\pi \frac{x}{\Lambda_1}\right) + a_2 \cos\left(2\pi \frac{x}{\Lambda_2}\right)$$

$$\frac{a_1}{2}\delta\left(u+\frac{1}{\Lambda_1}\right)\delta(v) + \frac{a_2}{2}\delta\left(u+\frac{1}{\Lambda_2}\right)\delta(v) +$$
$$\frac{a_1}{2}\delta\left(u-\frac{1}{\Lambda_1}\right)\delta(v) + \frac{a_2}{2}\delta\left(u-\frac{1}{\Lambda_2}\right)\delta(v)$$

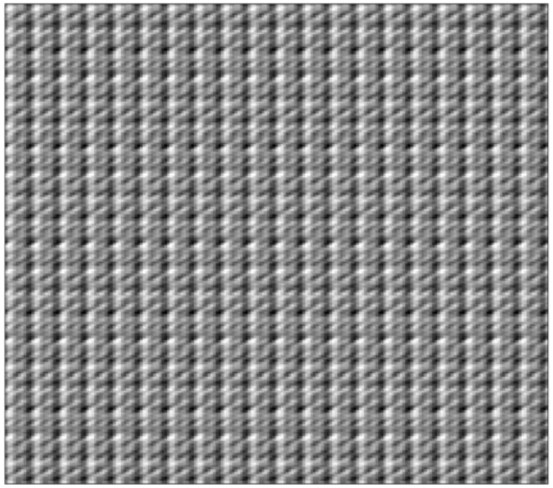


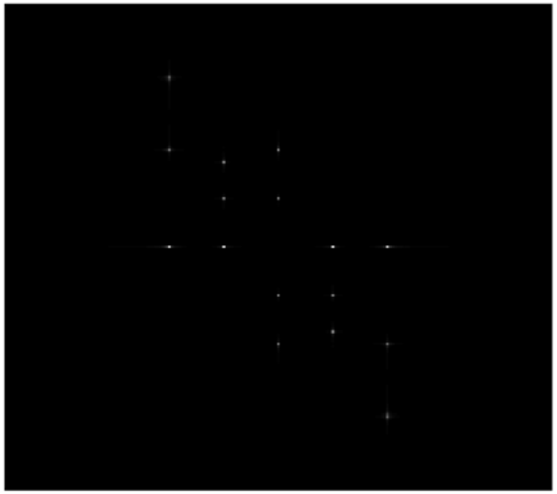


Space domain

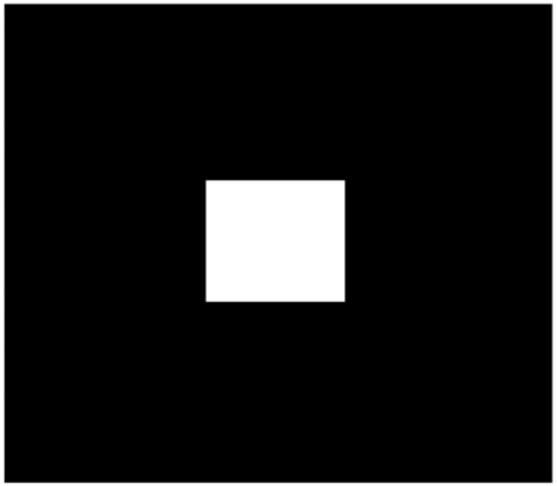
Frequency (Fourier) domain

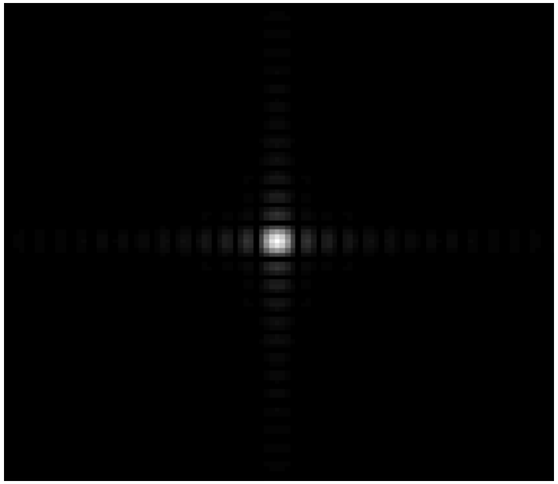
Superposition: multiple gratings





discrete
(Fourier
series)



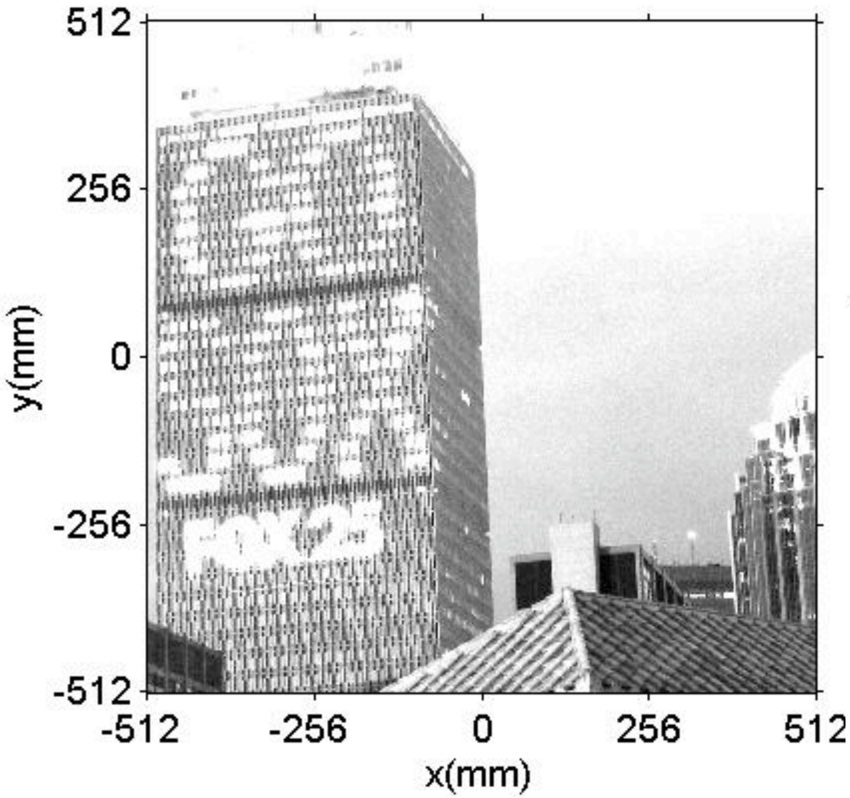


continuous
(Fourier
integral)

Space
domain

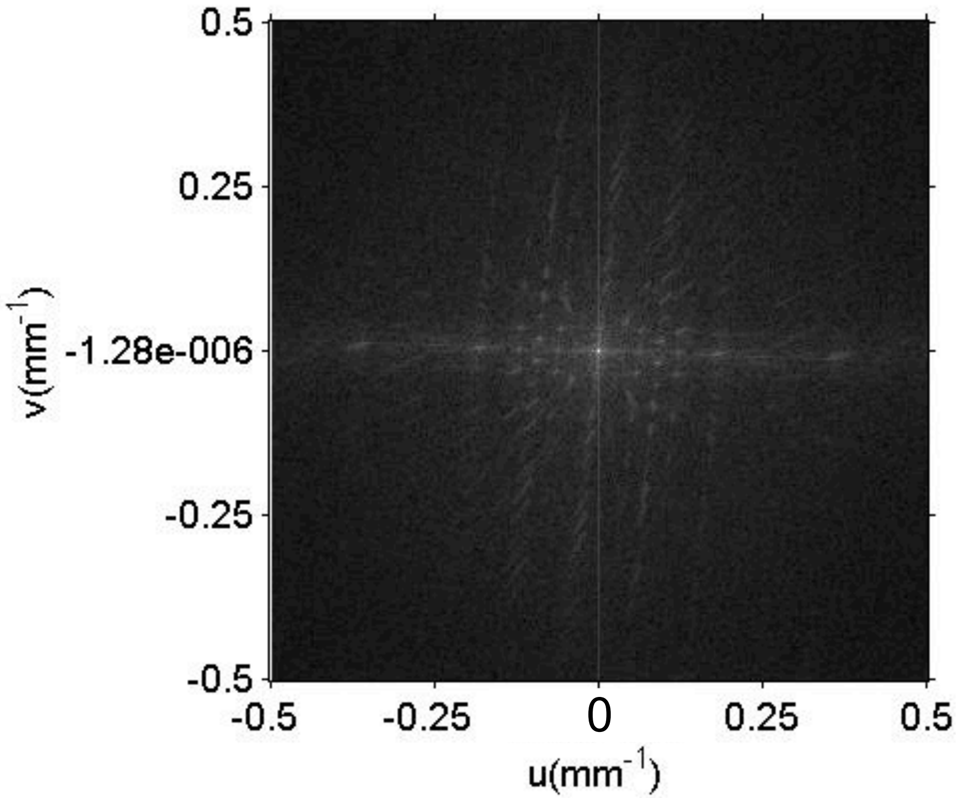
Frequency
(Fourier)
domain

Spatial frequency representation of arbitrary scenes



Space domain

$$g(x,y)$$

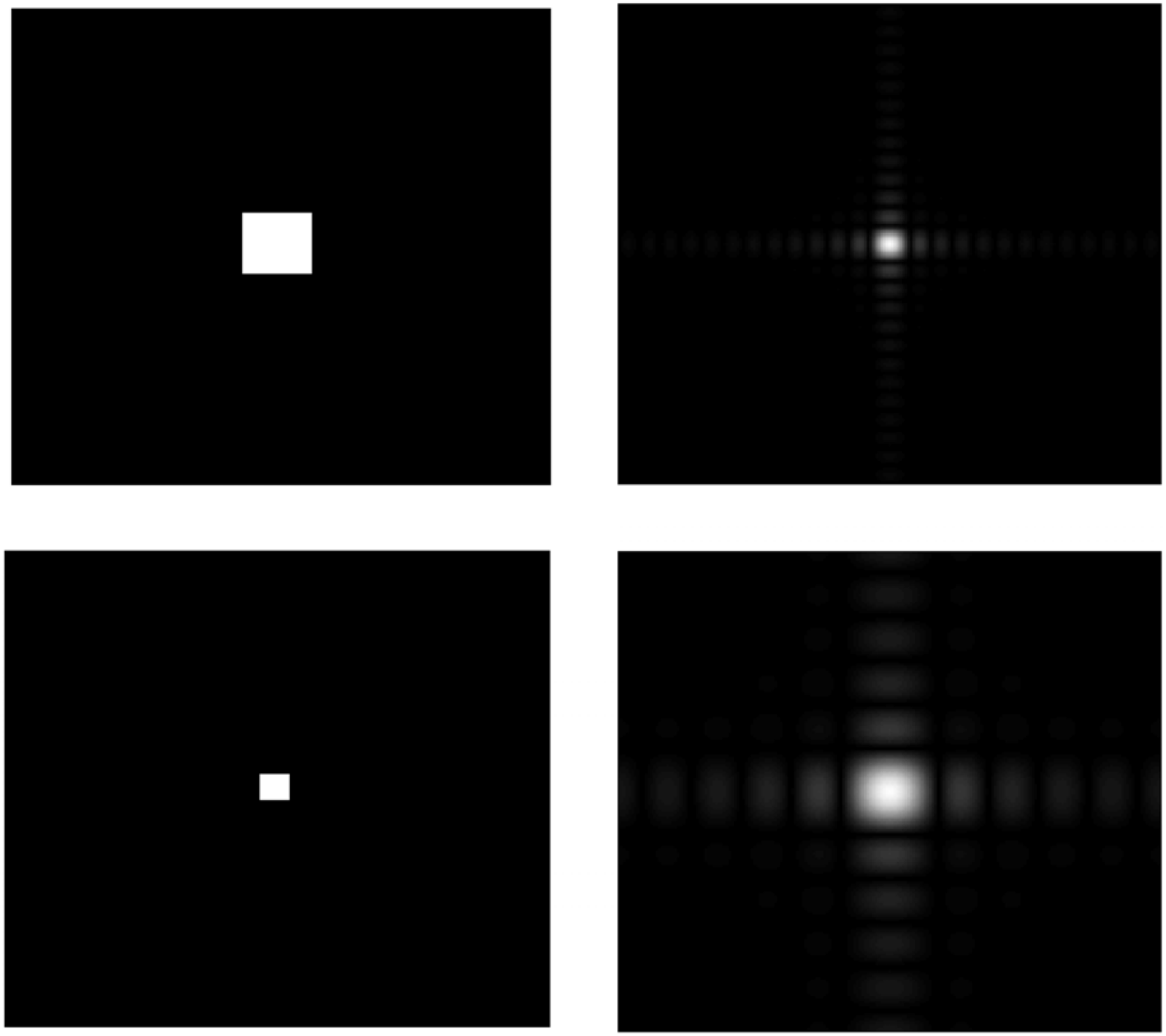


Spatial frequency domain

$$G(u,v) = \mathcal{F}\left\{g(x,y)\right\}$$

Fourier transform

The scaling (or similarity) theorem

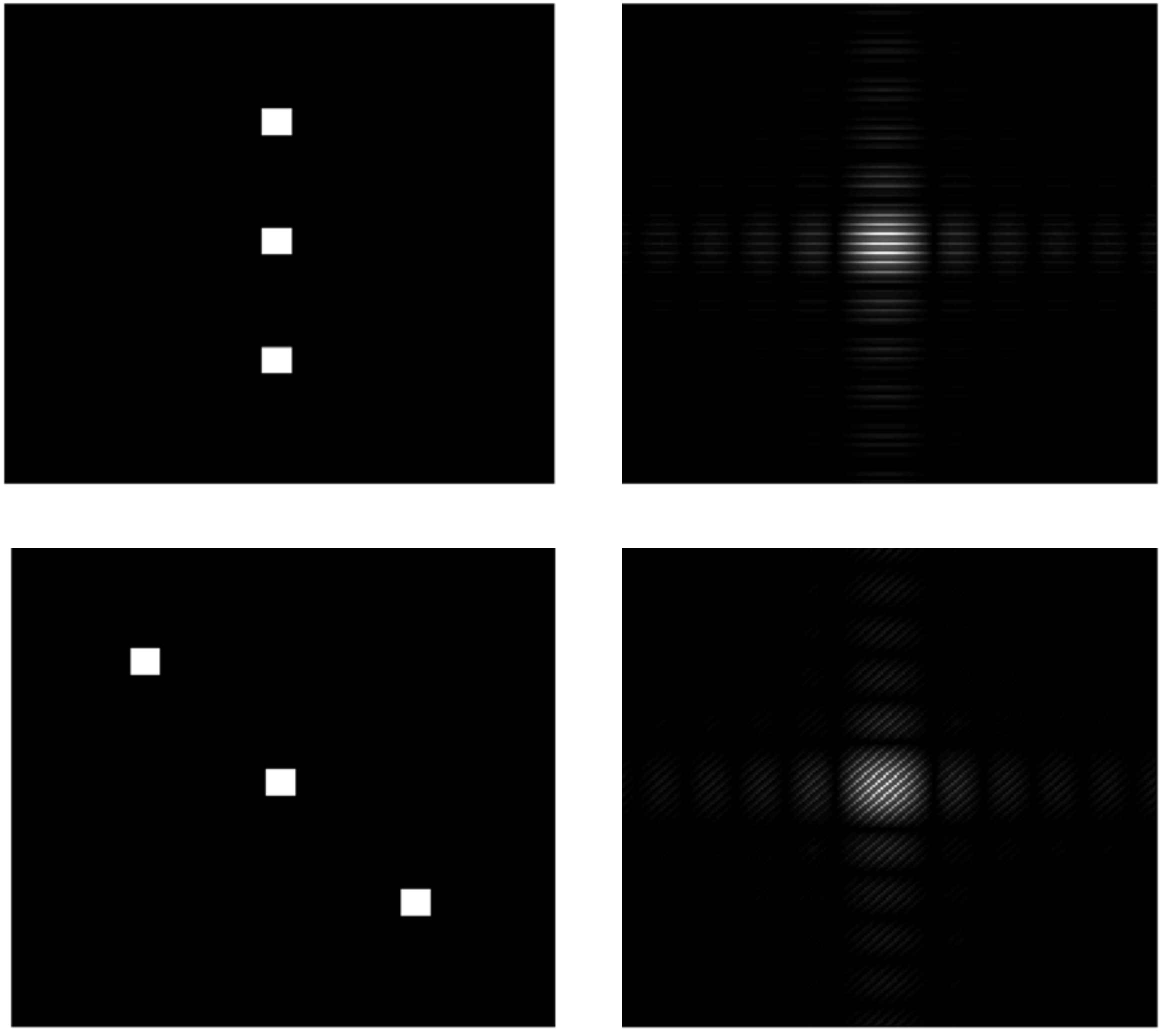


Space domain

Frequency (Fourier) domain

$$\mathcal{F}\left\{g\left(\frac{x}{a},\frac{y}{b}\right)\right\} = |ab|\,G(au, bv)$$

The shift theorem



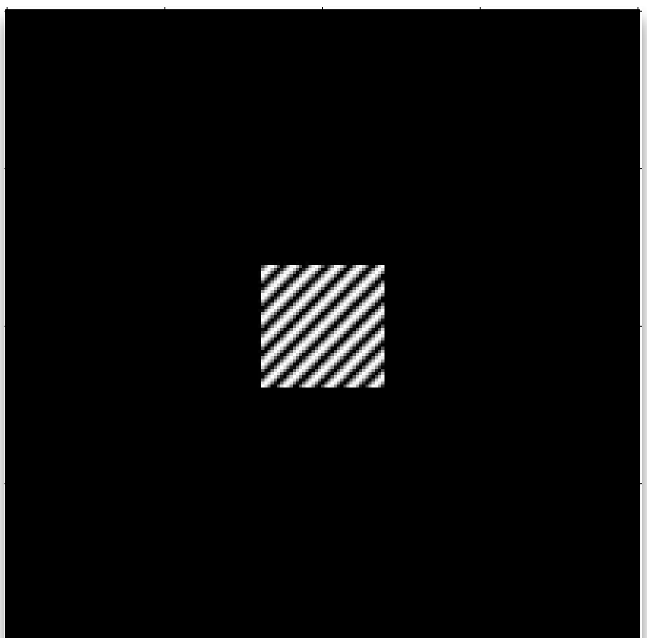
Space domain

Frequency (Fourier) domain

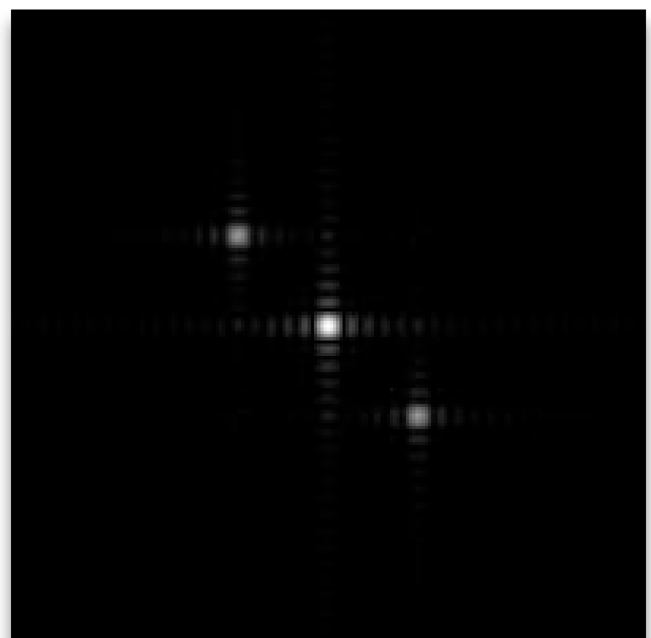
$$\mathcal{F}\left\{g\left(x-a,y-b\right)\right\}=\exp\left\{2\pi\left(au+bv\right)\right\}G(u,v)$$

The convolution theorem

multiplication



convolution



2.71 / 2.710 Optics
Spring 2009